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The Linear Cutwidth and Cyclic Cutwidth of Complete n-Partite Graphs

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THE LINEAR CUTWIDTH AND CYCLIC CUTWIDTH OF COMPLETE N-PARTITE GRAPHS

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

by

Stephanie Anne Creswell

June 2014

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ABSTRACT

The cutwidth of different graphs is a topic that has been extensively studied. The basis of this paper is the cutwidth of complete n -partite graphs. While looking at the cutwidth of complete n -partite graphs, we strictly consider the linear embedding and cyclic embedding. The relationship between the linear cutwidth and the cyclic cutwidth is discussed and used throughout multiple proofs of different cases for the cyclic cutwidth. All the known cases for the linear and cyclic cutwidth of complete bipartite, complete tripartite, and complete n -partite graphs are highlighted.

The main focus of this paper is to expand on the cyclic cutwidth of complete tripartite graphs. Using the relationship of the linear cutwidth and cyclic cutwidth of any graph, we find a lower bound and an upper bound for the cyclic cutwidth of complete tripartite graph $K_{r,r,pr}$ where r is odd and p is a natural number. Throughout this proof there are two cases that develop, p even and p odd. Within each case we have to consider the cuts of multiple regions to find the maximum cut of the cyclic embedding. Once all regions within each case are considered, we discover that the upper and lower bounds are equivalent. This discovery of the cyclic cutwidth of complete tripartite graph $K_{r,r,pr}$ where r is odd and p is a natural number results in getting one step closer to finding the cyclic cutwidth of any complete tripartite graph $K_{r,s,t}$.

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Chapter 1

Background

In this chapter we will introduce complete n -partite graphs and two different ways these graphs can be arranged. Within the linear embedding we will discuss the linear cutwidth of complete bipartite and complete tripartite graphs. We will only discuss the proofs involving the linear embedding and linear cutwidth of complete bipartite graphs, since the proofs for the complete tripartite graphs are similar. Within the cyclic embedding we will discuss certain cases discovered for the cyclic cutwidth of complete bipartite graphs and complete tripartite graphs. We will only discuss the proof of the lower bound for the cyclic cutwidth of any complete tripartite graph and the proof of the cyclic cutwidth for graph $K_{r,r,r}$ when r is odd since they relate to the proof in chapter 2.

1.1 Complete n -Partite Graphs

A graph consists of vertices and edges, where one edge connects two vertices. Visually, vertices are represented by dots and edges are represented by lines. Although there are many different types of graphs, the focus of this paper will be on complete n -partite graphs, denoted K_{m_1,m_2,\dots,m_n} . A complete n -partite graph is defined to have n different groups of vertices V_1, V_2, \dots, V_n , where $|V_1| = m_1, |V_2| = m_2, \dots, |V_n| = m_n$, such that no vertex within group V_k , $k = 1, 2, \dots, n$, connects to any other vertex within group V_k , but is connected to every vertex not in group V_k . Note $m_k \geq 1$ and $n \geq 2$.

Many examples and proofs within this paper will consist of complete bipartite and complete tripartite graphs. A complete bipartite graph is an n -partite graph where

$n = 2$. A complete tripartite graph is a complete n -partite graph where $n = 3$. In this paper a complete bipartite graph will be represented by $K_{m,n}$ and complete tripartite graph will be represented by $K_{r,s,t}$. Below Figure 1.1 shows examples of a complete bipartite graph $K_{3,4}$ and a complete tripartite graph $K_{2,3,4}$.

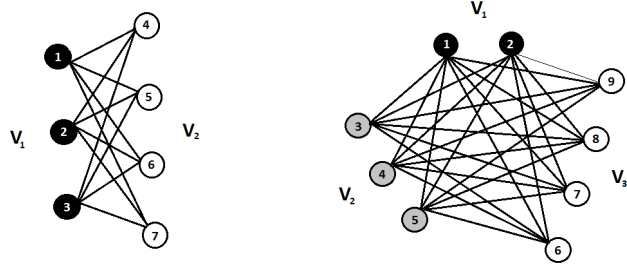


Figure 1.1: Complete Bipartite Graph $K_{3,4}$ and Complete Tripartite Graph $K_{2,3,4}$

Vertices of an n -partite graph can be arranged in many different embeddings. This paper will focus on a linear embedding and a cyclic embedding.

1.2 Linear Embedding

A linear embedding consists of an arrangement of all the vertices in a single line. Figure 1.2 is the linear embedding of the complete tripartite graph $K_{2,3,4}$ from Figure 1.1. The vertices that were connected in the non-linear representation of the graph will also be connected in this new linear embedding.

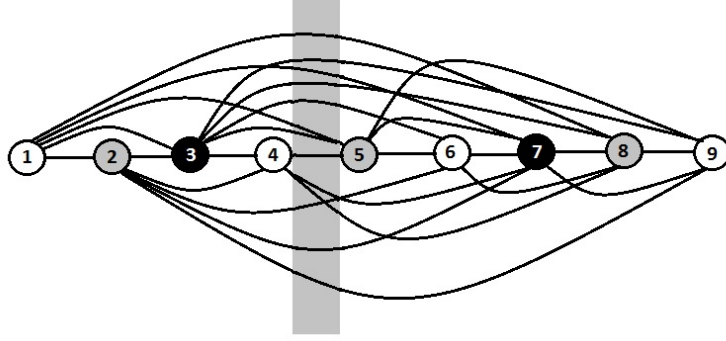


Figure 1.2: Linear Embedding of Complete Tripartite Graph $K_{2,3,4}$

A region in a linear embedding is the area between two adjacent vertices denoted $(v_1, v_2, \dots, x - (v_1 + v_2 + \dots + v_{(n-1)}))$ for K_{m_1, m_2, \dots, m_n} , where x represents the number of vertices to the left of the cut and v_k represents the number of vertices left of the cut from group V_k . Note that $x - (v_1 + v_2 + \dots + v_{(n-1)})$ is the number of vertices to the left of the cut from group V_n . In Figure 1.2 above, the region between vertices four and five is shaded and is denoted as $(1, 1, 2)$ where $x = 4$.

1.3 Cyclic Embedding

A cyclic embedding consists of an arrangement of all the vertices in a circle. Figure 1.3 is the cyclic embedding of the complete bipartite graph $K_{3,4}$ from Figure 1.1. The vertices that were connected in the non-cyclic representation of the graph will also be connected in this new cyclic embedding.

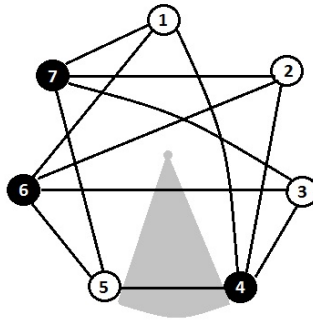


Figure 1.3: Cyclic Embedding of Complete Bipartite Graph $K_{3,4}$

A region in a cyclic embedding is the area of a sector from the center of the circle to two consecutive vertices on the edge of the circle. It is important to note that the cut of a region (discussed later) is dependent on how many edges cross through the region. Thus, no edge can pass through the center of the circle and must take a path through adjacent regions to connect two vertices. Above in Figure 1.3, the region between vertices four and five is shaded.

1.4 Cutwidth

The cut of a region is the number of edges that cross through the region between two consecutive vertices. The region with the maximum cut within an embedding is the region with the most edges crossing through it. Figure 1.4 shows the cut of each region for one possible arrangement of the linear embedding and cyclic embedding of the complete bipartite graph $K_{3,5}$. The maximum cut of the arrangement for the linear embedding is 9 and the maximum cut for the arrangement of the cyclic embedding is 5.

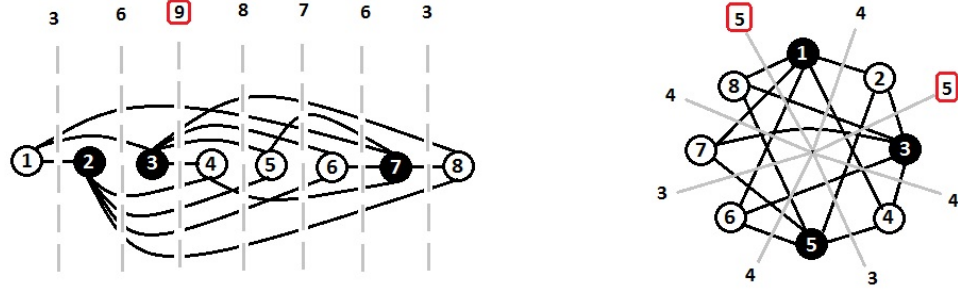


Figure 1.4: Cut of Each Region of Graph $K_{3,5}$ Within a Linear and Cyclic Embedding

Note that these maximum cuts may not be the smallest maximum cut for the linear embedding and the cyclic embedding for the graph $K_{3,5}$. There may be another arrangement within these embeddings that produce a smaller maximum cut. Figure 1.5 shows a second linear arrangement of the graph $K_{3,5}$ that reduces the maximum cut.

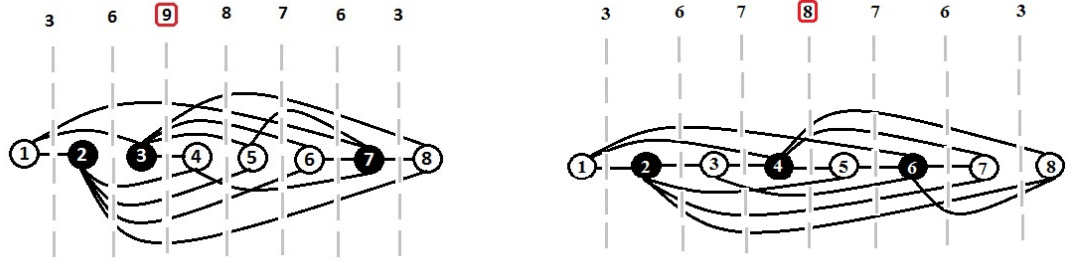


Figure 1.5: Cut of Each Region of Graph $K_{3,5}$ Within a Linear Embedding

The smallest maximum cut for all arrangements within each embedding is the cutwidth of a graph. How to find the linear cutwidth and cyclic cutwidth for complete n -partite graphs is the main focus of this paper.

1.5 Linear Cut

Recall a region in a linear embedding is the area between two adjacent vertices denoted $(v_1, v_2, \dots, x - (v_1 + v_2 + \dots + v_{(n-1)}))$ for K_{m_1, m_2, \dots, m_n} , where x represents the number of vertices to the left of the cut and v_k represents the number of vertices left of the cut from group V_k . The cut of a region in a linear embedding can be found by multiplying the number of vertices from each group left of the cut $v_i, i = 1, 2, \dots, n$ by the number of vertices from each group right of the cut $m_j - n_j, j = 1, 2, \dots, n$ and $j \neq i$. Thus, for a bipartite graph the cut of a region for a linear embedding can be found by using the following.

$$\text{cut}(v_1, x - v_1) = v_1(n - (x - v_1)) + (x - v_1)(m - v_1)$$

Similarly for a tripartite graph the cut of a region for a linear embedding can be found by using the following.

$$\begin{aligned} \text{cut}(v_1, v_2, x - (v_1 + v_2)) &= v_1[(s - v_2) + (t - (x - v_1 - v_2))] \\ &\quad + v_2[(r - v_1) + (t - (x - v_1 - v_2))] \\ &\quad + (x - v_1 - v_2)[(r - v_1) + (s - v_2)] \end{aligned}$$

1.6 Linear Cutwidth of Complete Bipartite Graphs

Before we look at the linear cutwidth of complete bipartite graphs we will look at an arrangement of the linear embedding that minimizes the cut of each region.

Theorem 1 ([Bow04]). *Let $K_{m,n}$ be a complete bipartite graph with two sets of vertices V_1 and V_2 , where $|V_1| = m$ and $|V_2| = n$, and let $m \leq n$. Let x represent the number of vertices to the left of a region. Then the cut of each region of a linear embedding for $K_{m,n}$ is minimized by placing $\frac{2x+m-n}{4}$ vertices from V_1 to the left of the cut.*

The following proof is a modified version of S. Bowles' proof. [Bow04]

Proof. Let x be the number of vertices to the left of region, $(v_1, x - v_1)$, within the linear embedding of a complete bipartite graph $K_{m,n}$. Suppose there are v_1 vertices from set

V_1 to the left of this region. Thus, there are $x - v_1$ vertices left of this region from set V_2 . Note there are $m - v_1$ vertices from set V_1 and $n - (x - v_1)$ vertices from set V_2 to the right of this region. The cut of this region is as follows.

$$\begin{aligned}
 \text{cut}(v_1, x - v_1) &= v_1(n - (x - v_1)) + (x - v_1)(m - v_1) \\
 &= v_1n - v_1x + v_1^2 + xm - v_1x - v_1m + v_1^2 \\
 &= 2v_1^2 + (n - 2x - m)v_1 + xm
 \end{aligned}$$

Let $f(v_1) = 2v_1^2 + (n - 2x - m)v_1 + xm$. Notice that $f(v_1)$ is a continuous function of $v_1 \in \mathbb{R}$, and $f(v_1) = \text{cut}(v_1, x - v_1)$ for $0 \leq v_1 \leq m$ and $v_1 \in \mathbb{Z}$. Note that the cut of a region will vary depending on the arrangement of the linear embedding. The different values for the cut of a region can be represented by a discrete function since we cannot have an edge partially intersect a region. However, the function $f(v_1)$ is a continuous representation of the possible cuts for a region and only the natural numbered solutions can be considered. Since $f(v_1)$ is a positive quadratic function it has a minimum value. If this minimum value is an integer, it represents the number of vertices from set V_1 to be are placed to the left of a region and will provide a minimum cut for that region. To find this minimum value, we take the derivative of $f(v_1)$ and set it equal to zero. Thus,

$$\begin{aligned}
 f'(v_1) &= 4v_1 + n - 2x - m \\
 0 &= 4v_1 + n - 2x - m \\
 \implies v_1 &= \frac{m+2x-n}{4}
 \end{aligned}$$

If v_1 is not an integer, then $f(v_1)$ is a positive quadratic function, we get this minimum value of $f(v_1)$ by rounding to the nearest whole number, denoted $[v_1]$. Note that this only happens since the graph is a positive quadratic. Thus, we are gaurenteed a minimum value that can be rounded to a whole number. Therefore, for every region of

the linear embedding of $K_{m,n}$, it's cut is minimized by placing $\lceil \frac{m+2x-n}{4} \rceil$ vertices from set V_1 to the left of each region. \square

Now that the linear embedding of a complete bipartite graph can be arranged such that the cut of each region is minimized, we will now look for the region of this arrangement that contains the maximum cut.

Corollary 1 ([Bow04]). *Let $K_{m,n}$ be a complete bipartite graph whose linear embedding is arranged by Theorem 1. Then the maximum cut will occur when $x = \frac{m+n}{2}$ for $m+n$ even and when $x = \frac{m+n-1}{2}$ and $x = \frac{m+n+1}{2}$ for $m+n$ odd. Also, the cuts to the left of the middle cut will be strictly increasing left to right and the cuts to the right of the middle cut will be strictly decreasing left to right.*

The following proof is a modified version of S. Bowles' proof. [Bow04]

Proof. Arrange the linear embedding by Theorem 1. Recall from the proof of Theorem 1, $\text{cut}(v_1, x - v_1) = 2v_1^2 + (n - 2x - m)v_1 + xm$. Also recall $v_1 = \frac{m+2x-n}{4}$, minimizes the cut of each region for the specific linear embedding. So consider the following.

$$\begin{aligned}
 \text{cut}(v_1, x - v_1) &= 2v_1^2 + (n - 2x - m)v_1 + xm \\
 &= 2v_1^2 - 4\frac{(m+2x-n)v_1}{4} + xm \\
 &= 2v_1^2 - 4v_1^2 + xm \\
 &= -2v_1^2 + xm \\
 &= -2\left(\frac{(m+2x-n)}{4}\right)^2 + xm \\
 &= -\frac{1}{8}(m+2x-n)^2 + xm
 \end{aligned}$$

Let $f(x) = -\frac{1}{8}(m+2x-n)^2 + xm$. Notice that $f(x)$ is a continuous function of $x \in \mathbb{R}$, and $f(x)$ gives the cut of each region for $1 \leq x \leq m+n$. The cut of each region

for the minimized arrangement can be represented by a discrete function since we cannot have an edge partially intersect a region. However, the function $f(x)$ is a continuous representation of the the cuts for each region for the minimized arrangement, and only the natural numbered solutions can be considered. Since $f(x)$ is a negative quadratic function it has a maximum value. If this maximum value is an integer, it represents the region where the maximum cut occurs for the minimized arrangement. To find this maximum value, we take the derivative of $f(x)$ and set it equal to zero. Thus,

$$f'(x) = -\frac{1}{4}(m + 2x - n)2 + m$$

$$= \frac{m+n}{2} - x$$

$$0 = \frac{m+n}{2} - x$$

Observe that $\frac{m+n}{2} - x > 0$ when $1 < x < \frac{m+n}{2}$ and $f(x)$ is increasing. Also $\frac{m+n}{2} - x < 0$ when $\frac{m+n}{2} < x < m+n$ and $f(x)$ is decreasing. Thus, for $m+n$ even, the maximum cut occurs at $x = \frac{m+n}{2}$. However, when $m+n$ is odd, $\frac{m+n}{2}$ is not an integer but is equally spaced between $\frac{m+n-1}{2}$ and $\frac{m+n+1}{2}$. Thus, the maximum cut will occur at $x = \frac{m+n-1}{2}$ and $x = \frac{m+n+1}{2}$. \square

The proof of the linear cutwidth of a complete bipartite graph follows from Theorem 1 and Corollary 1.

Theorem 2 ([Bow04]). *Let $K_{m,n}$ be a complete bipartite graph. Then,*

$$lcw(K_{m,n}) = \begin{cases} \frac{mn}{2} & mn \text{ even} \\ \frac{mn+1}{2} & mn \text{ odd} \end{cases}$$

The following proof is a modified version of S. Bowles' proof. [Bow04]

Proof. Let the complete bipartite graph $K_{m,n}$ be arranged by the algorithm in Theorem 1. From Theorem 1 and Corollary 1 the linear cutwidth of $K_{m,n}$ occurs when $x = \frac{m+n}{2}$ for $m+n$ even, and when $x = \frac{m+n-1}{2}$ and $x = \frac{m+n+1}{2}$ for $m+n$ odd. To find the linear cutwidth of $K_{m,n}$, we need to consider both cases.

Case 1: $m + n$ even

Let $m + n$ be even, then the middle region of the linear embedding is at $x = \frac{m+n}{2}$. By Theorem 1 place $v_1 = \lceil \frac{2x+m-n}{4} \rceil$ vertices from set V_1 to the left of the middle region. Substituting $\frac{m+n}{2}$ for x , we get $v_1 = \lceil \frac{m+n+m-n}{4} \rceil = \lceil \frac{m}{2} \rceil$. Thus, $v_1 = \frac{m}{2}$ when m is even and $v_1 = \frac{m+1}{2}$ when m is odd. Since $m + n$ is even, we know that when m is even n is also even, and when m is odd n is also odd. Therefore, the number of vertices, v_2 , from set V_2 to the left of the middle region are as follows.

$$v_2 = \begin{cases} x - \frac{m}{2} = \frac{n}{2} & \text{for } n \text{ even} \\ x - \frac{m+1}{2} = \frac{n-1}{2} & \text{for } n \text{ odd} \end{cases}$$

Recall from section 1.5, the cut of a region can be found by $cut(v_1, x - v_1) = v_1(n - (x - v_1)) + (x - v_1)(m - v_1)$. Consider the cut when m and n are both even.

$$\begin{aligned} cut\left(\frac{m}{2}, \frac{n}{2}\right) &= \frac{m}{2} \frac{n}{2} + \frac{m}{2} \frac{n}{2} \\ &= \frac{mn}{2} \end{aligned}$$

Now consider the cut when m and n are both odd.

$$\begin{aligned} cut\left(\frac{m+1}{2}, \frac{n-1}{2}\right) &= \frac{m+1}{2} \frac{n-1}{2} + \frac{m-1}{2} \frac{n+1}{2} \\ &= \frac{mn+1}{2} \end{aligned}$$

Case 2: $m + n$ odd

Let $m + n$ be odd, then the middle regions of the linear embedding are at $x = \frac{m+n+1}{2}$ and $x = \frac{m+n-1}{2}$. Without loss of generality, let's consider the region where $x = \frac{m+n-1}{2}$. By Theorem 1 place $v_1 = \lceil \frac{2x+m-n}{4} \rceil$ vertices from set V_1 to the left of the middle region. Substituting $\frac{m+n-1}{2}$ for x , we get $v_1 = \lceil \frac{m+n-1+m-n}{4} \rceil = \lceil \frac{2m-1}{4} \rceil = \lceil \frac{m}{2} - \frac{1}{4} \rceil$. Thus, $v_1 = \frac{m}{2}$ when m is even and $v_1 = \frac{m-1}{2}$ when m is odd. Since $m + n$ is odd, we know that when m is even n is odd, and when m is odd n is even. Therefore, the number of vertices, v_2 , from set V_2 to the left of the middle region are as follows.

$$v_2 = \begin{cases} x - \frac{m}{2} = \frac{n-1}{2} & \text{for } n \text{ odd} \\ x - \frac{m-1}{2} = \frac{n}{2} & \text{for } n \text{ even} \end{cases}$$

Consider the cut when m is even and n is odd.

$$\begin{aligned} cut\left(\frac{m}{2}, \frac{n-1}{2}\right) &= \frac{m}{2} \frac{n+1}{2} + \frac{m}{2} \frac{n-1}{2} \\ &= \frac{mn}{2} \end{aligned}$$

Now consider the cut when m is odd and n is even.

$$\begin{aligned} cut\left(\frac{m-1}{2}, \frac{n}{2}\right) &= \frac{m-1}{2} \frac{n}{2} + \frac{m+1}{2} \frac{n}{2} \\ &= \frac{mn}{2} \end{aligned}$$

Therefore,

$$lcw(K_{m,n}) = \begin{cases} \frac{mn}{2} & mn \text{ even} \\ \frac{mn+1}{2} & mn \text{ odd} \end{cases}$$

□

1.7 Linear Cutwidth of Complete Tripartite Graphs

To find the linear cutwidth of a complete tripartite graph, the same structure of proofs of a complete bipartite graph was followed. As a result Theorem 1, Corollary 1, and Theorem 2 were developed. Note that the middle region consists of an arrangement of r vertices from each set V_1 , V_2 , and V_3 , where r represents the number of vertices from the smallest set. The outer region is the arrangement of the remaining vertices and is evenly distributed on either side of the middle region.

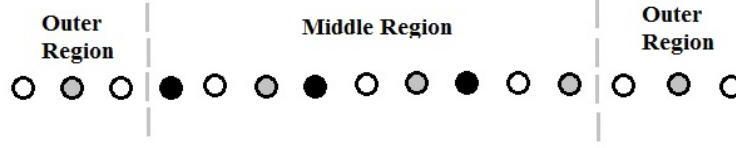


Figure 1.6: Tripartite Arrangement of Graph $K_{3,5,7}$

Theorem 3 ([Bow04]). Let $K_{r,s,t}$ be a complete tripartite graph with three sets of vertices V_1 , V_2 , and V_3 , where $|V_1| = r$, $|V_2| = s$, and $|V_3| = t$, and let x represent the number of vertices to the left of a region. To minimize each cut of the linear embedding for $K_{r,s,t}$, the middle and outer sections of the graph are minimized independently. The middle cuts are minimized by placing $\frac{2x+2r-s-t}{6}$ vertices from A , $\frac{2x+2s-r-t}{6}$ vertices from B , $\frac{2x+2t-r-s}{6}$ vertices from C to the left of each cut. The outer sections are minimized according to Theorem 1 for complete bipartite graphs.

Corollary 2 ([Bow04]). Let $K_{r,s,t}$ be a complete tripartite graph whose linear embedding is arranged by Theorem 3. Then the maximum cut will occur when $x = \frac{r+s+t}{2}$ for $r+s+t$ even and when $x = \frac{r+s+t-1}{2}$ and $x = \frac{r+s+t+1}{2}$ for $r+s+t$ odd.

Theorem 4 ([Bow04]). Let $K_{r,s,t}$ be a complete tripartite graph. Then,

$$lcw(K_{r,s,t}) = \begin{cases} \frac{rs+rt+st}{2} & \text{for two or more } s, r, t \text{ even} \\ \frac{rs+rt+st+1}{2} & \text{otherwise} \end{cases}$$

As a result, the linear embedding of complete bipartite and tripartite graphs can be arranged in such a way that the cut of each region can be minimized. From this the region with the maximum cut can be located and the cut itself can be found. These discoveries were used to relate the linear embedding with the cyclic embedding, which in return resulted in multiple cases for the cyclic cutwidth of complete bipartite and tripartite graphs.

1.8 Lowerbound for Cyclic Cutwidth

The following relationship has been made between linear cutwidth and cyclic cutwidth for any graph G .

Theorem 5 ([Joh03]). *For any graph G ,*

$$ccw(G) \geq \frac{lcw(G)}{2}$$

The following proof is a modified version of M. Johnson's proof. [Joh03]

Proof. Consider the cyclic embedding of graph G . Number the vertices clockwise from a_1 to a_n , where n represents the number of vertices in graph G . Let a_1 be the vertex immediately clockwise to the region where the cyclic cutwidth occurs. Let x represent the cyclic cutwidth. Also let the cut of the region immediately counter-clockwise to vertex a_i be α_i for all $i = 1, 2, \dots, n$.

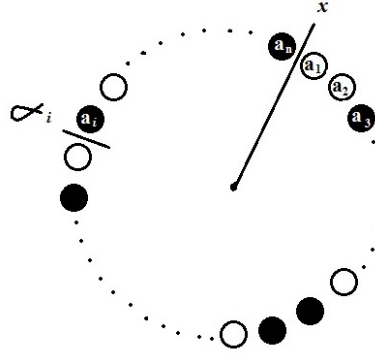
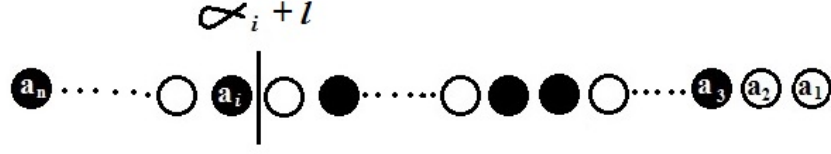


Figure 1.7: Cyclic Arrangement of Graph G

Now take the cyclic embedding above and place the same arrangement of vertices in a linear embedding such that vertex a_n is the most left vertex and vertex a_1 is the most right vertex. Let the region from the cyclic embedding with cut α_i be the same region within the linear embedding where the maximum cut occurs. Let y represent this linear cutwidth. Note the cut of this region within the linear embedding will increase by some number l . Thus, the linear cutwidth of the linear embedding is $y = \alpha_i + l$.

Figure 1.8: Linear Arrangement of Graph G

Recall x is the cyclic cutwidth and $y = \alpha_i + l$ is the linear cutwidth of any graph G . Assume $x < \frac{y}{2}$. Since $l < x$ we know $\alpha_i + l \leq \alpha_i + x$. Also since $\alpha_i < x$ we can claim $\alpha_i + l \leq 2x$. So by the hypothesis, $\alpha_i + l \leq y$. However, this is a contradiction since $\alpha_i + l$ is strictly equal to y . Thus, $x \geq \frac{y}{2}$, which implies the following.

$$ccw(G) \geq \frac{lcw(G)}{2}$$

□

Theorem 5 is used to prove multiple algorithms for finding the cyclic cutwidth of complete bipartite and tripartite graphs. In the following sections we will only look at some of the proofs since all of these proofs follow a similar structure.

1.9 Cyclic Cutwidth of Complete Bipartite Graphs

Unlike the linear embedding, one algorithm to find the cyclic cutwidth for any complete bipartite graph has yet to be discovered. However, there have been multiple discoveries to find the cyclic cutwidth of complete bipartite graphs with different restrictions. Below is a list of the known discoveries.

Theorem 6 ([Hol03] [Joh03]). *Let $K_{m,n}$ be a complete bipartite graph. Then,*

$$ccw(K_{m,n}) = \begin{cases} \frac{mn}{4} & m, n \text{ both even} \\ \frac{mn+3}{4} & m = n \text{ both odd} \\ \frac{mn+j}{4} & m \text{ odd}, j = \frac{n}{m}, j \text{ even} \\ \frac{mn+j+2}{4} & m \text{ odd}, j = \frac{n}{m}, j \text{ odd} \\ \frac{mn+2}{4} & m \equiv 2(\text{mod } 4), n \text{ odd}, 2n \geq m \\ \frac{mn+4}{4} & m \equiv 0(\text{mod } 4), n \text{ odd}, 2n \geq m \\ \frac{mn+l+2}{4} & m \text{ even}, n \text{ odd}, 2n < m, l \text{ even}, m - ln < 2n, m - ln \equiv 2(\text{mod } 4) \\ \frac{mn+l+4}{4} & m \text{ even}, n \text{ odd}, 2n < m, l \text{ even}, m - ln < 2n, m - ln \equiv 0(\text{mod } 4) \\ \frac{mn+l+2}{4} & m \text{ and } n \text{ odd}, m < n, l \text{ odd}, n - lm < 2m, n - lm \equiv 2(\text{mod } 4) \end{cases}$$

Even though many cases to find the cyclic cutwidth of complete bipartite graphs have been proven, there are still other cases to be considered and they can be very tedious. The different cases still to be considered will be discussed later. Thus, some territory for the cyclic cutwidth of complete tripartite graphs has been explored. The results show similarities to the cyclic cutwidth of complete bipartite graphs.

1.10 Cyclic Cutwidth of Complete Tripartite Graphs

A lower and upper bound for the cyclic cutwidth has been proven for any complete tripartite graph. Lemma 1 has been used to find the cyclic cutwidth of complete tripartite graphs with different restrictions shown in the following theorems.

Lemma 1 ([All06]). *Let $K_{r,s,t}$ be a complete tripartite graph. Then, the lower bound for the cyclic cutwidth of $K_{r,s,t}$ is,*

$$ccw(K_{r,s,t}) \geq \frac{rs+rt+st}{4}$$

The following proof is a modified version of H. Allmond's proof. [All06]

Proof. Recall by Theorem 5, for any graph G we have the following.

$$ccw(G) \geq \frac{lcw(G)}{2}$$

Also recall by Theorem 4, for any complete tripartite graph the linear cutwidth is as follows.

$$lcw(K_{r,s,t}) = \begin{cases} \frac{rs+rt+st}{2} & \text{for two or more } s, r, t \text{ even} \\ \frac{rs+rt+st+1}{2} & \text{otherwise} \end{cases}$$

Since we are looking for a lower bound we will only consider $\frac{rs+rt+st}{2}$. Thus, we can claim the following is a lower bound for the cyclic cutwidth for any complete tripartite graph $K_{r,s,t}$.

$$ccw(K_{r,s,t}) \geq \frac{rs+rt+st}{4}$$

□

Lemma 2 ([All06]). *Let $K_{r,s,t}$ be a complete tripartite graph. Then, the upper bound for the cyclic cutwidth of $K_{r,s,t}$ is,*

$$ccw(K_{r,s,t}) \leq ccw(K_{r,s}) + ccw(K_{r+s,t})$$

Theorem 7 and Theorem 8 are two different cases that have been proven for the cyclic cutwidth of complete tripartite graphs.

Theorem 7 ([All06]). *Let $K_{r,s,t}$ be a complete tripartite graph such that r, s, t are all even. Then,*

$$ccw(K_{r,s,t}) = \frac{rs+rt+st}{4}$$

Theorem 8 ([All06]). *Let $K_{r,r,r}$ be a complete tripartite graph. Then, for r odd,*

$$ccw(K_{r,r,r}) = \frac{3r^2+1}{4}$$

The following proof was inspired by H. Allmond's proof. [All06]

Proof. To show the cyclic cutwidth of $K_{r,r,r}$ is as stated above we need to show the lower bound and upper bound are equivalent. Consider the following.

Lower Bound

By Lemma 1, a lower bound of the cyclic cutwidth for any complete tripartite graph $K_{r,s,t}$ is as follows.

$$ccw(K_{r,s,t}) \geq \frac{rs+rt+st}{4}$$

Using Lemma 1 consider the complete tripartite graph $K_{r,r,r}$ where r is odd. The lower bound is as follows.

$$\begin{aligned} ccw(K_{r,r,r}) &\geq \frac{rr+rr+rr}{4} \\ &\geq \frac{3r^2}{4} \end{aligned}$$

However, $\frac{3r^2}{4}$ is not an integer. Since r is odd, to round to the next integer let $r = 2m + 1$ where $m = 1, 2, 3, \dots$. Then

$$\begin{aligned} 3r^2 &= 3(2m+1)^2 \\ &= 12m^2 + 12m + 3 \\ &\equiv 3 \pmod{4} \end{aligned}$$

So $3r^2 + 1 \equiv 0 \pmod{4}$, hence $4 \mid (3r^2 + 1)$. So to round up to the next integer we would need to add 1 to the numerator. Thus, the lower bound for the cyclic cutwidth of $K_{r,r,r}$ with r odd is,

$$ccw(K_{r,r,r}) \geq \frac{3r^2+1}{4}$$

Upper Bound

The following arrangement will be used to come up with an upperbound. Within the arrangement we are going to look for a region that has the maximum cut. Let V_1 consist of r black vertices, V_2 consist of r grey vertices, and V_3 consist of r white vertices. We will make r groups of vertices consisting of one black vertex, one grey vertex and one white vertex. Within each cyclic embedding the vertices will be arranged clockwise starting with a black vertex at the top followed by one white vertex, then one grey vertex. This pattern will continue until all $3r$ vertices have been placed. This arrangement minimizes the number of regions we need to consider for the maximum cut. Since each set of black, grey, and white vertices is a repeated pattern, the cuts for each similar region will also be repeated, so we only need to consider the different regions within one set of black, grey, and white vertices. Thus, we only need to consider one region where the maximum cut may lie, the region immediately clockwise to the black vertex call it α_1 . The remaining regions will be similar to region α_1 and will be discussed further after the upperbound of the cyclic cutwidth is found for region α_1 . Figure 1.9 is an example of this arrangement for $K_{5,5,5}$.

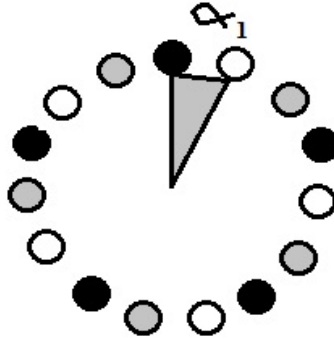


Figure 1.9: Vertex Arrangement for $K_{5,5,5}$

For graph $K_{r,r,r}$, r being odd results in the total number of vertices to be odd, thus there will be no diameters to consider. Therefore, all edges will take the shortest route within the cyclic embedding to connect two vertices. This will ensure each cut to be minimized so that not one or more regions are overloaded with edges. To get the cut of region α_1 we will look at the vertices whose edges contribute to the cut. We will look at them in the following order: the top black vertex connecting to the grey and white vertices, the remaining black vertices connecting to the grey and white vertices, and the grey vertices connecting to the white vertices.

Let's first consider the edges connecting the top black vertex to the grey and white vertices that will contribute to the cut of region α_1 . Since the total number of vertices is odd, each vertex will be directly across from a region. Thus, there would be a combination of r white and grey vertices clockwise from the top black vertex to the opposite region and the remaining r white and grey vertices counter-clockwise from the top black vertex to the opposite region. The grey and white vertices counter-clockwise from the top black vertex to the opposite region will not contribute any edges to the cut of region α_1 . However, each grey and white vertex clockwise from the top black vertex to the opposite region will contribute an edge to the cut. Thus, the top black vertex will contribute a total of r edges to the cut of region α_1 . As you can see in Figure 1.10 only the grey and white vertices clockwise from the top black vertex to the opposite region will contribute an edge to the cut of region α_1 .

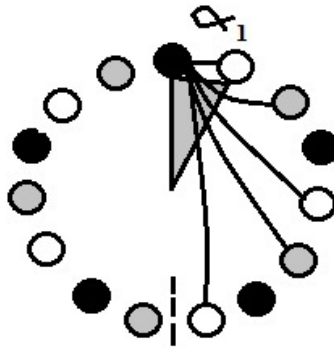


Figure 1.10: Top Black Vertex Edge Contribution to α_1 of $K_{5,5,5}$

We now need to consider the remaining $r - 1$ black vertices. There will be $\frac{r-1}{2}$ black vertices clockwise from the top black vertex to its opposite region and $\frac{r-1}{2}$ black vertices counter-clockwise from the top black vertex to its opposite region. Also, there will be a combination of r grey and white vertices clockwise from a black vertex to its opposite region and the remaining r grey and white vertices counter-clockwise from the same black vertex to its opposite region. Therefore, between each pair of consecutive black vertices there will be one grey vertex and one white vertex. Thus, each black vertex clockwise and counter-clockwise from the top black vertex to its opposite region will have a multiple of two grey and white vertices between the top black vertex and the black vertex being considered. As you can see in Figure 1.11 there are two black vertices clockwise and counter-clockwise from the top black vertex to its opposite region. There is one grey vertex and one white vertex between each black vertex.

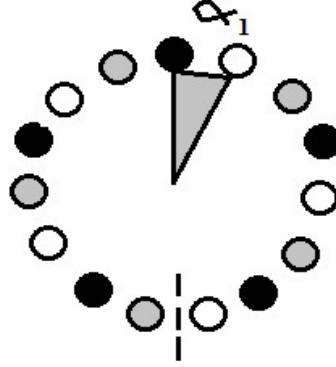


Figure 1.11: Remaining $r - 1$ Black Vertices Arrangement of $K_{5,5,5}$

Let's consider the first black vertex clockwise from the top black vertex. The grey and white vertices clockwise to this black vertex to its opposite region will not contribute any edges to the cut of region α_1 . So consider the remaining r grey and white vertices counter-clockwise. There will be two grey and white vertices between this black vertex and the top black vertex that also will not contribute any edges to the cut of region α_1 . However, the remaining $r - 2$ grey and white vertices will each contribute an edge to the cut of region α_1 . See Figure 1.12 for an example.

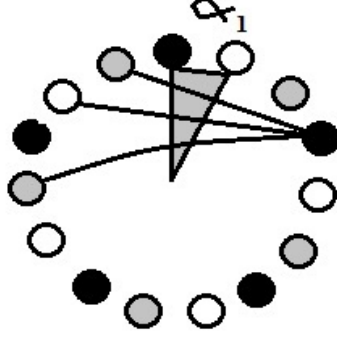


Figure 1.12: First Clockwise Black Vertex Edge Contribution to α_1 of $K_{5,5,5}$

Now consider the second black vertex clockwise from the top black vertex. Similarly the grey and white vertices clockwise to this black vertex to its opposite region will not contribute any edges to the cut of region α_1 . Considering the remaining r grey and white vertices counter-clockwise, there will be four grey and white vertices between this black vertex and the top black vertex that will also not contribute any edges to the cut of region α_1 . Therefore, the remaining $r - 4$ grey and white vertices will each contribute an edge to the cut of region α_1 .

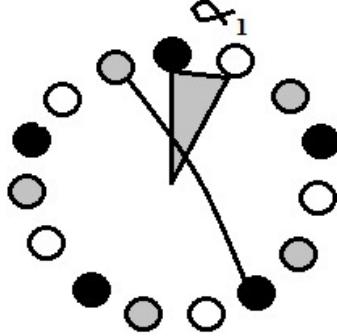


Figure 1.13: Second Clockwise Black Vertex Edge Contribution to α_1 of $K_{5,5,5}$

This pattern will continue until we reach the region opposite the top black vertex. Thus, the total edges contributing to the cut of region α_1 from each $\frac{r-1}{2}$ black vertex clockwise from the top black vertex will be as follows.

$$\begin{aligned}
[r-2] + [r-4] + \dots + [r-2\left(\frac{r-1}{2}\right)] &= \sum_{n=1}^{\frac{r-1}{2}} [r-2n] \\
&= \sum_{n=1}^{\frac{r-1}{2}} r - 2 \sum_{n=1}^{\frac{r-1}{2}} n \\
&= r\left(\frac{r-1}{2}\right) - 2\left(\frac{\frac{r-1}{2}\left(\frac{r-1}{2}+1\right)}{2}\right) \\
&= \frac{r^2-2r+1}{4}
\end{aligned}$$

Considering the other $\frac{r-1}{2}$ black vertices counter-clockwise from the top black vertex, the same pattern will occur contributing another $\frac{r^2-2r+1}{4}$ edges to the cut of region α_1 .

Now we need to consider the edges connecting the grey vertices to the white vertices that will contribute to the cut of region α_1 . There will be $\frac{r-1}{2}$ grey vertices clockwise from the top black vertex to its opposite region and $\frac{r+1}{2}$ grey vertices counter-clockwise from the top black vertex to its opposite region. Let's consider the grey vertices clockwise to the top black vertex first. For each grey vertex clockwise to the top black vertex to the region opposite it there are $\frac{r-1}{2}$ white vertices clockwise to the grey vertex being considered that will not contribute an edge to the cut of region α_1 . However, there are $\frac{r+1}{2}$ white vertices counter-clockwise to each of these grey vertices that need to be considered. So let's look at the first grey vertex clockwise to the top black vertex. There is one white vertex between this grey vertex and the top black vertex that will not contribute an edge to the cut of region α_1 . However, the remaining $\frac{r+1}{2} - 1$ white vertices counter-clockwise to this grey vertex will each contribute an edge to the cut of region α_1 .

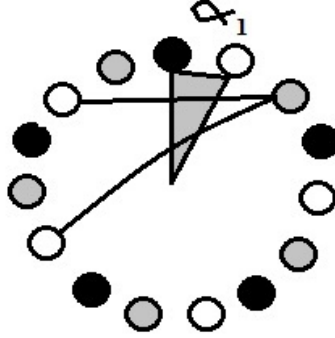


Figure 1.14: First Clockwise Grey Vertex Edge Contribution to α_1 of $K_{5,5,5}$

Now consider the second grey vertex clockwise to the top grey vertex. Note between consecutive grey vertices there is one white vertex. Also between any two grey vertices there is a multiple of of white vertices. So considering the second grey vertex clockwise to the top black vertex, we now have two white vertices between the top black vertex and the second grey vertex that will not contribute any edges to the cut of region α_1 . However, the remaining $\frac{r+1}{2} - 2$ white vertices counter-clockwise to this grey vertex will each contribute an edge to the cut of region α_1 .

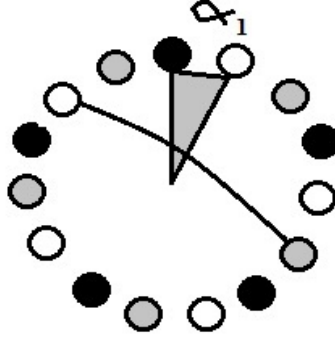


Figure 1.15: Second Clockwise Grey Vertex Edge Contribution to α_1 of $K_{5,5,5}$

This pattern will continue until the last grey vertex just before the region opposite the top black vertex is reached. Thus, the total edges contributing to the cut of region α_1 from each $\frac{r-1}{2}$ grey vertex clockwise from the top black vertex will be as follows.

$$\begin{aligned}
& \left[\frac{r+1}{2} - 1\right] + \left[\frac{r+1}{2} - 2\right] + \dots + \left[\frac{r+1}{2} - \frac{r-1}{2}\right] \\
&= \sum_{n=1}^{\frac{r-1}{2}} \left[\frac{r+1}{2} - n\right] \\
&= \frac{1}{2} \left[\sum_{n=1}^{\frac{r-1}{2}} r + \sum_{n=1}^{\frac{r-1}{2}} 1 \right] - \sum_{n=1}^{\frac{r-1}{2}} n \\
&= \frac{1}{2} \left[r \left(\frac{r-1}{2} \right) + \frac{r-1}{2} \right] - \left(\frac{\frac{r-1}{2} \left(\frac{r-1}{2} + 1 \right)}{2} \right) \\
&= \frac{r^2 - 1}{8}
\end{aligned}$$

Now consider the other $\frac{r+1}{2}$ grey vertices counter-clockwise from the top black vertex to the region opposite of it. Note, for each grey vertex counter-clockwise to the top black vertex to the region opposite it there are now $\frac{r+1}{2}$ white vertices counter-clockwise to the grey vertex being considered that will not contribute any edges to the cut of region α_1 . However, there are $\frac{r-1}{2}$ white vertices clockwise to each of these grey vertices that need to be considered. So let's look at the first grey vertex counter-clockwise to the top black vertex. Now there are no white vertices between this grey vertex and the top black vertex. Thus, all $\frac{r-1}{2}$ white vertices clockwise to this grey vertex will each contribute an edge to the cut of region α_1 .

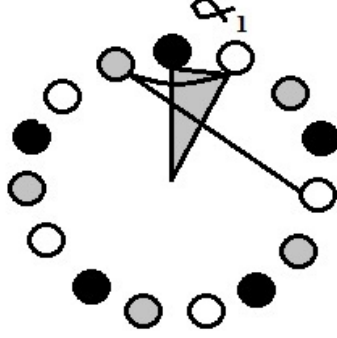


Figure 1.16: First Counter-Clockwise Grey Vertex Edge Contribution to α_1 of $K_{5,5,5}$

Also between consecutive grey vertices there is still one white vertex, and between any two grey vertices there is still a multiple of white vertices. So a pattern similar to the other grey vertices occurs. Thus, the total edges contributing to the cut of region α_1 from each $\frac{r+1}{2}$ grey vertex clockwise from the top black vertex will be as follows.

$$\begin{aligned}
 & \left\lceil \frac{r-1}{2} \right\rceil + \left\lceil \frac{r-1}{2} - 1 \right\rceil + \left\lceil \frac{r-1}{2} - 2 \right\rceil + \dots + \left\lceil \frac{r-1}{2} - \left(\frac{r+1}{2} - 1 \right) \right\rceil \\
 &= \sum_{n=1}^{\frac{r+1}{2}} \left\lceil \frac{r-1}{2} - (n-1) \right\rceil \\
 &= \frac{1}{2} \left[\sum_{n=1}^{\frac{r+1}{2}} r - \sum_{n=1}^{\frac{r+1}{2}} 1 \right] - \sum_{n=1}^{\frac{r+1}{2}} n + \sum_{n=1}^{\frac{r+1}{2}} 1 \\
 &= \frac{1}{2} \left[r \left(\frac{r+1}{2} \right) - \frac{r+1}{2} \right] - \left[\frac{\frac{r+1}{2} \left(\frac{r+1}{2} + 1 \right)}{2} \right] + \frac{r+1}{2} \\
 &= \frac{r^2-1}{8}
 \end{aligned}$$

Now that all possible edges contributing to the cut of region α_1 have been considered, the total edges contributing to the cut are as follows.

$$\begin{aligned}
cut(\alpha_1) &= r + 2 \left(\frac{r^2 - 2r + 1}{4} \right) + 2 \left(\frac{r^2 - 1}{8} \right) \\
&= \frac{3r^2 + 1}{4}
\end{aligned}$$

Now let's consider the other regions within the set of one black vertex, one grey vertex, and one white vertex that will have a similar cut to the region α_1 . The cut of the region immediately clockwise to the white vertex and the cut of the region immediately clockwise to the grey vertex will be equal to the cut of region α_1 since all the regions are symmetrical.

Since the cut of all regions for this cyclic arrangement have been evaluated, we claim the upperbound for the cyclic cutwidth of $K_{r,r,r}$ for r odd is,

$$ccw(K_{r,r,r}) \leq \frac{3r^2 + 1}{4}$$

Also, since the upper bound and the lower bound are the same, we claim the cyclic cutwidth of $K_{r,r,r}$ for r odd is,

$$ccw(K_{r,r,r}) = \frac{3r^2 + 1}{4}$$

□

Theorem 7 and Theorem 8 only scratch the surface of cyclic cutwidth of complete tripartite graphs. In Chapter 2 we explore a new case for the cyclic cutwidth of complete tripartite graphs that is similar to Theorem 8.

Chapter 2

Cyclic Cutwidth of Complete Tripartite Graph $K_{r,r,pr}$ For r Odd and p a Natural Number

In this chapter we are going to explore the cyclic cutwidth of complete tripartite graph $K_{r,r,pr}$, where r is odd and p is a natural number. In order to find the cyclic cutwidth we will find a lower and upper bound and show that they match. When looking at both bounds two cases develop, when p is even and when p is odd.

Theorem 9. *For a complete tripartite graph $K_{r,r,pr}$ where r is odd and p a natural number, the cyclic cutwidth is,*

$$ccw(K_{r,r,pr}) = \begin{cases} \frac{(2p+1)r^2+1}{4} & \text{for } p \text{ odd} \\ \frac{(2p+1)r^2+3}{4} & \text{for } p \text{ even} \end{cases}$$

Proof. To show the cyclic cutwidth of $K_{r,r,pr}$ is as stated above we need to show the lower bound and upper bound are equivalent. Consider the following.

2.1 Lower Bound

By lemma 1, a lower bound of the cyclic cutwidth for any complete tripartite graph $K_{r,s,t}$ is as follows.

$$ccw(K_{r,s,t}) \geq \frac{rs+rt+st}{4}$$

Using lemma 1 consider the complete tripartite graph $K_{r,r,pr}$ where r is odd and p a natural number. The lower bound is as follows.

$$\begin{aligned} ccw(K_{r,r,pr}) &\geq \frac{rr+r(pr)+r(pr)}{4} \\ &\geq \frac{(2p+1)r^2}{4} \end{aligned}$$

However, $\frac{(2p+1)r^2}{4}$ is not an integer. There are two cases when rounding up to the next integer, p being odd and p being even.

2.1.1 Case 1: p is Odd

p is odd. Let $p = 2n + 1$ where $n = 1, 2, 3, \dots$. Then,

$$\begin{aligned} 2p + 1 &= 2(2n + 1) + 1 \\ &= 4n + 3 \\ &\equiv 3 \pmod{4} \end{aligned}$$

Since r is odd, let $r = 2m + 1$ where $m = 1, 2, 3, \dots$. Then,

$$\begin{aligned}
(2p+1)r^2 &\equiv 3r^2 \pmod{4} \\
&\equiv 3(2m+1)^2 \pmod{4} \\
&\equiv 3(4m^2+4m+1) \pmod{4} \\
&\equiv 3 \pmod{4}
\end{aligned}$$

Thus, $(2p+1)r^2+1 \equiv 0 \pmod{4}$ hence $4|(2p+1)r^2+1$. Therefore, to round up to the next integer we would need to add one to the numerator. Thus, when p is odd a lower bound for the cyclic cutwidth of $K_{r,r,pr}$ with r odd is,

$$ccw(K_{r,r,pr}) \geq \frac{(2p+1)r^2+1}{4}$$

2.1.2 Case 2: p is Even

p is even. Let $p = 2n$ where $n = 1, 2, 3, \dots$. Then,

$$\begin{aligned}
2p+1 &= 2(2n)+1 \\
&= 4n+1 \\
&\equiv 1 \pmod{4}
\end{aligned}$$

Since r is odd, again let $r = 2m+1$ where $m = 1, 2, 3, \dots$. Then,

$$\begin{aligned}
(2p+1)r^2 &\equiv r^2 \pmod{4} \\
&\equiv (2m+1)^2 \pmod{4} \\
&\equiv 4m^2 + 4m + 1 \pmod{4} \\
&\equiv 1 \pmod{4}
\end{aligned}$$

Thus, $(2p+1)r^2 + 3 \equiv 0 \pmod{4}$ hence $4|(2p+1)r^2 + 3$. Therefore, to round up to the next integer we would need to add 3 to the numerator. Thus, when p is even a lower bound for the cyclic cutwidth of $K_{r,r,pr}$ with r odd is,

$$ccw(K_{r,r,pr}) \geq \frac{(2p+1)r^2+3}{4}$$

Therefore, for a complete tripartite graph $K_{r,r,pr}$ where r is odd and p a natural number, a lower bound of the cyclic cutwidth is,

$$ccw(K_{r,r,pr}) \geq \begin{cases} \frac{(2p+1)r^2+1}{4} & \text{for } p \text{ odd} \\ \frac{(2p+1)r^2+3}{4} & \text{for } p \text{ even} \end{cases}$$

2.2 Upper Bound

A specific arrangement will be used to come up with an upperbound. Within the arrangement we are going to look for a region that has the maximum cut. Let V_1 consist of r black vertices, V_2 consist of r grey vertices, and V_3 consist of pr white vertices. We will make r groups of vertices consisting of one black vertex, one grey vertex and p white vertices. Within each cyclic embedding the vertices will be arranged clockwise starting with a black vertex at the top followed by $\frac{p}{2}$ white vertices, then a grey vertex followed by another $\frac{p}{2}$ white vertices. If p is even then $\frac{p}{2}$ is an integer and p white vertices will be evenly distributed between consecutive black and grey vertices. If p is odd then $\frac{p}{2}$ is not an integer. In this case we will place $\frac{p+1}{2}$ white vertices immediately clockwise to

the black vertex and $\frac{p-1}{2}$ white vertices immediately clockwise to the grey vertex. This pattern will continue until all $(p+2)r$ vertices have been placed. Figure 2.1 is an example of this arrangement for $K_{3,3,6}$ and $K_{3,3,9}$.

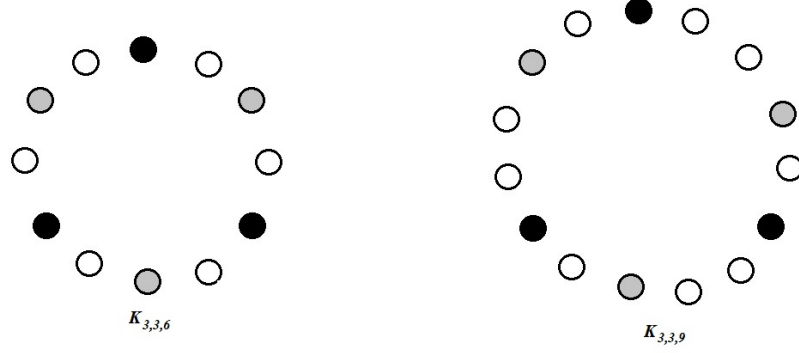


Figure 2.1: Vertex Arrangement for $K_{r,r,pr}$

This arrangement minimizes the number of regions we need to consider for the maximum cut. Since each set of black, grey and white vertices is a repeated pattern, the cuts for each similar region will also be repeated. Thus, we only need to consider the different regions within one set of black, grey, and white vertices.

As each region is considered for the upper bound of the cyclic cutwidth two cases develop, when p is even and when p is odd. When p is even, we need to consider diameters and how that will affect the maximum cut. When p is odd, there will be no diameters to consider; however, the white vertices will no longer be grouped evenly between each black and grey vertex. In either case the non-diameter edges will take the shortest route within the cyclic embedding to connect two vertices. This will ensure each cut to be minimized such that not one or more regions are overloaded with edges. So consider the following two cases.

2.2.1 Case 1: p is Even

Let p be even. Recall when p is even the arrangement will consist of one black vertex followed by $\frac{p}{2}$ white vertices, followed by one grey vertex, followed by $\frac{p}{2}$ white

vertices and then the pattern repeats. With this case we need to consider two different regions where the maximum cut may lie, the region immediately clockwise to the black vertex, call it α_1 , and the region immediately clockwise of the first white vertex, call it α_2 . The remaining regions will be similar to either region α_1 or region α_2 and will be discussed further after an upperbound of the cyclic cutwidth is found for region α_1 and region α_2 .

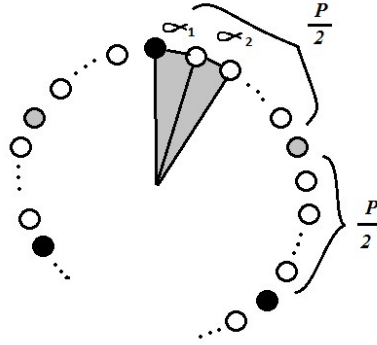
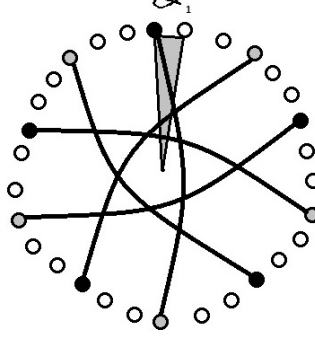


Figure 2.2: Regions α_1 and α_2

Consider region α_1 . To get the cut of this region we will look at the vertices whose edges contribute to the cut. We will look at them in the following order: diameters, the top black vertex connecting to the grey and white vertices, the remaining black vertices connecting to the grey and white vertices, and the grey vertices connecting to the white vertices.

Let's first consider the diameters. In this arrangement for every black vertex there will be one grey vertex directly across from it within the cyclic embedding. Also directly across from a white vertex there will always be another white vertex. Since white vertices cannot connect to white vertices, there will only be r diameters connecting each black vertex to the one grey vertex directly across from it. Since the diameters cannot travel straight through the center of the cyclic embedding, we will alternate each diameter clockwise then counter-clockwise around the center as shown in Figure 2.3. This will minimize the number of diameters crossing through each region. Therefore, either $\frac{r-1}{2}$ diameters or $\frac{r+1}{2}$ diameters will be contributing to the cut of each region. Since we are considering the maximum cut we will assume the cut of region α_1 contains $\frac{r+1}{2}$ diameters.

Figure 2.3: Diameters of $K_{5,5,20}$

Now let's consider the edges connecting the top black vertex to the grey and white vertices that will contribute to the cut of region α_1 . Since the edge connecting the top black vertex to the bottom grey vertex directly across from it is included in the diameters we will consider the grey and white vertices clockwise from the top black vertex to the bottom grey vertex and the grey and white vertices counter-clockwise from the top black vertex to the bottom grey vertex. There will be $\frac{pr}{2}$ white vertices clockwise from the top black vertex to the bottom grey vertex and $\frac{pr}{2}$ white vertices counter-clockwise from the top black vertex to the bottom grey vertex. There will also be $\frac{r-1}{2}$ grey vertices clockwise from the top black vertex to the bottom grey vertex and $\frac{r-1}{2}$ grey vertices counter-clockwise from the top black vertex to the bottom grey vertex. The grey and white vertices counter-clockwise from the top black vertex to the bottom grey vertex will not contribute any edges to the cut of region α_1 . However, each grey and white vertex clockwise from the top black vertex to the bottom grey vertex will contribute an edge to the cut. Thus, the top black vertex will contribute a total of $\frac{pr}{2} + \frac{r-1}{2}$ edges to the cut of region α_1 . As you can see in Figure 2.4 only the grey and white vertices clockwise from the top black vertex to the bottom grey vertex will contribute an edge to the cut of region α_1 .

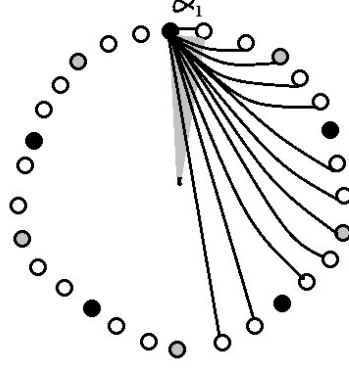


Figure 2.4: Top Black Vertex Edge Contribution to α_1 of $K_{5,5,20}$

We now need to consider the remaining $r - 1$ black vertices. There will be $\frac{r-1}{2}$ black vertices clockwise from the top black vertex to the bottom grey vertex and $\frac{r-1}{2}$ black vertices counter-clockwise from the top black vertex to the bottom grey vertex. Recall that each black vertex will be directly across from a grey vertex and will be connected by a diameter. Since the edges connecting a black vertex to its opposite grey vertex were already considered with the diameters they will not be counted here. Thus, there will be $\frac{pr}{2}$ white vertices clockwise from a black vertex to its opposite grey vertex and $\frac{pr}{2}$ white vertices counter-clockwise from the same black vertex to its opposite grey vertex. As well there will be $\frac{r-1}{2}$ grey vertices clockwise from the same black vertex to its opposite grey vertex and $\frac{r-1}{2}$ grey vertices counter-clockwise from the same black vertex to its opposite grey vertex. Therefore, between each pair of consecutive black vertices there will be a set of $p + 1$ grey and white vertices. Thus, each black vertex clockwise and counter-clockwise from the top black vertex to the bottom grey vertex will have a multiple of $p + 1$ grey and white vertices between the top black vertex and the black vertex being considered. As you can see in Figure 2.5 there are two black vertices clockwise and counter-clockwise from the top black vertex to the bottom grey vertex. There is a total of five grey and white vertices between each black vertex, one grey vertex and four white vertices. Consider the black vertex 3 in Figure 2.5. There are two sets of grey and white vertices between the top black vertex and black vertex 3, making a total of ten grey and white vertices between the two black vertices.

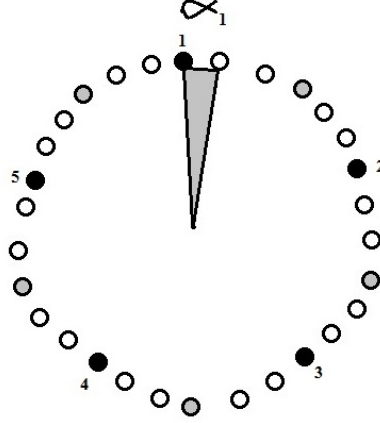


Figure 2.5: Remaining $r - 1$ Black Vertices Arrangement of $K_{5,5,20}$

Let's consider the first black vertex clockwise from the top black vertex. The grey and white vertices clockwise to this black vertex to the grey vertex directly across from it will not contribute any edges to the cut of region α_1 . So consider the remaining $\frac{pr}{2} + \frac{r-1}{2}$ grey and white vertices counter-clockwise. There will be $p + 1$ grey and white vertices between this black vertex and the top black vertex that also will not contribute any edges to the cut of region α_1 . However, the remaining $\frac{pr}{2} + \frac{r-1}{2} - (p + 1)$ grey and white vertices will each contribute an edge to the cut of region α_1 . See Figure 2.6 for an example.

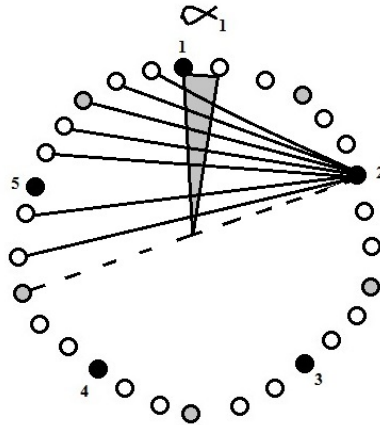


Figure 2.6: First Clockwise Black Vertex Edge Contribution to α_1 of $K_{5,5,20}$

Now consider the second black vertex clockwise from the top black vertex. Similarly the grey and white vertices clockwise to this black vertex to the grey vertex directly across from it will not contribute any edges to the cut of region α_1 . Considering the remaining $\frac{pr}{2} + \frac{r-1}{2}$ grey and white vertices counter-clockwise, there will be $2(p+1)$ grey and white vertices between this black vertex and the top black vertex that will also not contribute any edges to the cut of region α_1 . Therefore, the remaining $\frac{pr}{2} + \frac{r-1}{2} - 2(p+1)$ grey and white vertices will each contribute an edge to the cut of region α_1 . See Figure 2.7 for an example.

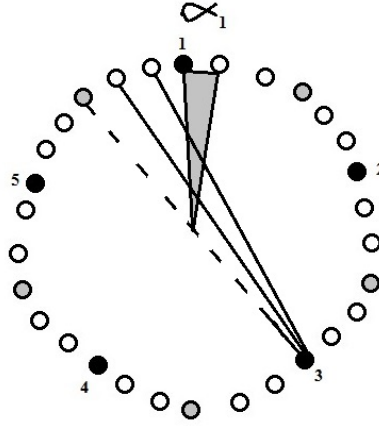


Figure 2.7: Second Clockwise Black Vertex Edge Contribution to α_1 of $K_{5,5,20}$

This pattern will continue until we reach the last black vertex just before the bottom grey vertex. Thus, the total edges contributing to the cut of region α_1 from each $\frac{r-1}{2}$ black vertex clockwise from the top black vertex will be as follows.

$$\begin{aligned}
& \left[\frac{pr}{2} + \frac{r-1}{2} - (p+1) \right] + \left[\frac{pr}{2} + \frac{r-1}{2} - 2(p+1) \right] + \dots + \frac{p}{2} \\
&= \frac{(p+1)(r-2)-1}{2} + \frac{(p+1)(r-4)-1}{2} + \dots + \frac{(p+1)(r-(r-1))-1}{2} \\
&= \sum_{n=1}^{\frac{r-1}{2}} \frac{(p+1)(r-2n)-1}{2} \\
&= \frac{p+1}{2} \left[\sum_{n=1}^{\frac{r-1}{2}} r - 2 \sum_{n=1}^{\frac{r-1}{2}} n \right] - \sum_{n=1}^{\frac{r-1}{2}} \frac{1}{2} \\
&= \frac{p+1}{2} \left[\left(\frac{r-1}{2} \right) r - 2 \left(\frac{\left(\frac{r-1}{2} \right) \left(\frac{r-1}{2} + 1 \right)}{2} \right) \right] - \frac{1}{2} \left(\frac{r-1}{2} \right) \\
&= \frac{(p+1)(r-1)^2}{8} - \frac{r-1}{4}
\end{aligned}$$

Considering the other $\frac{r-1}{2}$ black vertices counter-clockwise from the top black vertex, the same pattern will occur contributing another $\frac{(p+1)(r-1)^2}{8} - \frac{r-1}{4}$ edges to the cut of region α_1 .

Now we need to consider the edges connecting the grey vertices to the white vertices that will contribute to the cut of region α_1 . First consider the bottom grey vertex. The white vertices clockwise from the bottom grey vertex to the top black vertex will contribute no edges to the cut of region α_1 as shown in Figure 2.8. Similarly the white vertices counter-clockwise from the bottom grey vertex to the top black vertex will contribute no edges to the cut of region α_1 . Thus, we only need to consider the remaining $r-1$ grey vertices.

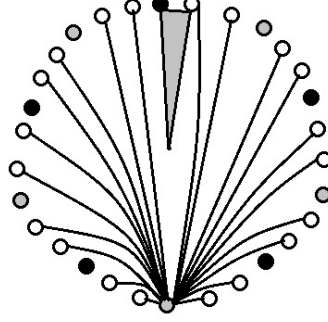
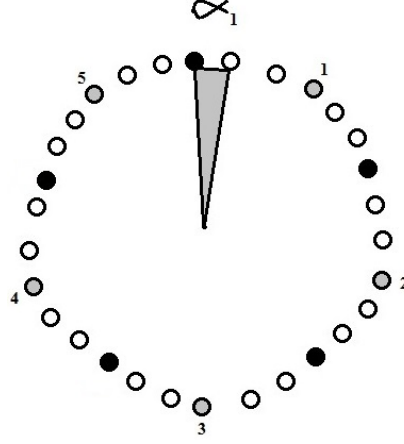
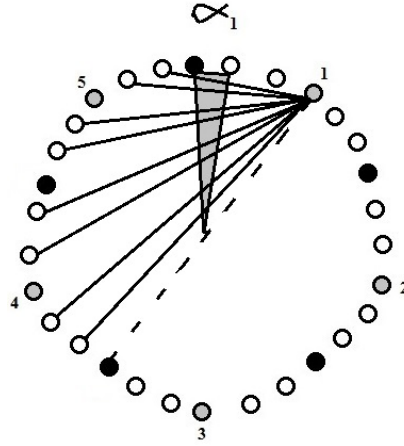


Figure 2.8: Bottom Grey Vertex Edge Contribution to α_1 of $K_{5,5,20}$

There will be $\frac{r-1}{2}$ grey vertices clockwise from the top black vertex to the bottom grey vertex and $\frac{r-1}{2}$ grey vertices counter-clockwise from the top black vertex to the bottom grey vertex. Recall that each grey vertex will be directly across from a black vertex. Thus, there will be $\frac{pr}{2}$ white vertices clockwise to a grey vertex to its opposite black vertex and $\frac{pr}{2}$ white vertices counter-clockwise to the same grey vertex to its opposite black vertex. Therefore, between each pair of consecutive grey vertices there will be a set of p white vertices and between each pair of consecutive grey and black vertices there will be a set of $\frac{p}{2}$ white vertices. Thus, each grey vertex clockwise and counter-clockwise from the top black vertex to the bottom grey vertex will have a multiple of $\frac{p}{2}$ white vertices between the top black vertex and the grey vertex being considered. As you can see in Figure 2.9 there are two grey vertices clockwise and counter-clockwise from the top black vertex to the bottom grey vertex. Also there are two white vertices between consecutive grey and black vertices and four white vertices between consecutive grey vertices. For example, grey vertex 2 has three sets of two white vertices between the top black vertex and itself.

Figure 2.9: Grey vertex Placement of $K_{5,5,20}$

Let's consider the first grey vertex clockwise from the top black vertex. The white vertices clockwise of this grey vertex to the black vertex directly across from it will not contribute any edges to the cut of region α_1 . So consider the remaining $\frac{pr}{2}$ white vertices counter-clockwise. There will be $\frac{p}{2}$ white vertices between this grey vertex and the top black vertex that also will not contribute any edges to the cut of region α_1 . However, the remaining $\frac{pr}{2} - \frac{p}{2}$ white vertices will each contribute an edge to the cut of region α_1 . See Figure 2.10 for an example.

Figure 2.10: First Clockwise Grey Vertex Edge Contribution to α_1 of $K_{5,5,20}$

Let's consider the second grey vertex clockwise from the top black vertex. The

white vertices clockwise to this grey vertex to the black vertex directly across from it will not contribute any edges to the cut of region α_1 . So consider the remaining $\frac{pr}{2}$ white vertices counter-clockwise. There will be $3\left(\frac{p}{2}\right)$ white vertices between this grey vertex and the top black vertex that also will not contribute any edges to the cut of region α_1 . However, the remaining $\frac{pr}{2} - 3\left(\frac{p}{2}\right)$ white vertices will each contribute an edge to the cut of region α_1 . See Figure 2.11 for an example.

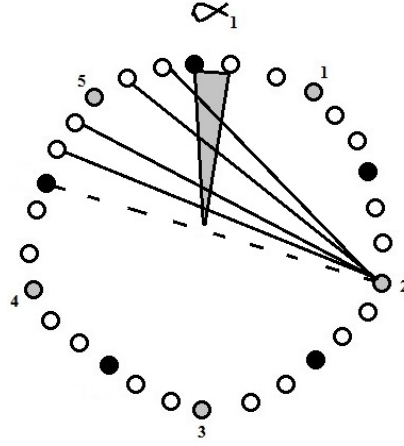


Figure 2.11: Second Clockwise Grey Vertex Edge Contribution to α_1 of $K_{5,5,20}$

This pattern will continue until we reach the last grey vertex just before the bottom grey vertex. Thus, the total edges contributing to the cut of region α_1 from each $\frac{r-1}{2}$ grey vertices clockwise from the top black vertex will be as follows.

$$\begin{aligned}
& \left\lfloor \frac{pr}{2} - \frac{p}{2} \right\rfloor + \left\lfloor \frac{pr}{2} - \frac{3p}{2} \right\rfloor + \left\lfloor \frac{pr}{2} - \frac{5p}{2} \right\rfloor + \dots + 0 \\
&= \frac{p(r-1)}{2} + \frac{p(r-3)}{2} + \dots + \frac{p(r-r)}{2} \\
&= \sum_{n=1}^{\frac{r+1}{2}} \frac{p[r-(2n-1)]}{2} \\
&= \frac{p}{2} \left[\sum_{n=1}^{\frac{r+1}{2}} r - 2 \sum_{n=1}^{\frac{r+1}{2}} n + \sum_{n=1}^{\frac{r+1}{2}} 1 \right] \\
&= \frac{p}{2} \left[r \left(\frac{r+1}{2} \right) - 2 \left(\frac{\left(\frac{r+1}{2} \right) \left(\frac{r+1}{2} + 1 \right)}{2} \right) + \left(\frac{r+1}{2} \right) \right] \\
&= \frac{p(r+1)(r-1)}{8}
\end{aligned}$$

Considering the other $\frac{r-1}{2}$ grey vertices counter-clockwise from the top black vertex the same pattern will occur contributing another $\frac{p(r+1)(r-1)}{8}$ edges to the cut of region α_1 .

Now that all possible edges contributing to the cut of region α_1 have been considered, the total edges contributing to the cut are as follows.

$$\begin{aligned}
cut(\alpha_1) &= \frac{r(p+1)-1}{2} + \frac{r+1}{2} + 2 \left[\frac{(p+1)(r-1)^2}{8} - \frac{r-1}{4} \right] + 2 \left[\frac{p(r+1)(r-1)}{8} \right] \\
&= \frac{(2p+1)r^2+3}{4}
\end{aligned}$$

Now let's consider the other cuts within the set of one black vertex, one grey vertex, and p white vertices that will have a similar cut to the region α_1 . The cut of the region immediately counter-clockwise to the black vertex will be less than or equal to the cut of region α_1 . Since the two regions are symmetrical, the non-diameter edges

will contribute an equivalent number of edges to the cut. The edges from diameters will contribute the same number of edges or one edge less than the region adjacent to it. Since the number of black vertices is equivalent to the number of grey vertices and the arrangement of the cyclic embedding is symmetrical, the region immediately clockwise and counter-clockwise to the grey vertex will also be less than or equal to the cut of region α_1 for the same reasons.

Now that an upper bound has been found for the region α_1 and the regions similar to it, we now need to consider the remaining regions. So consider region α_2 .

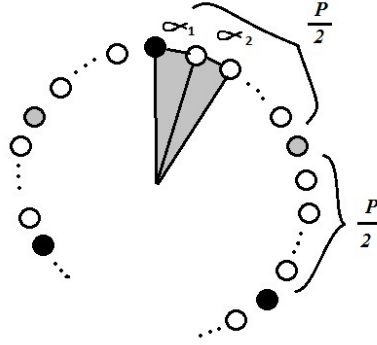


Figure 2.12: Regions α_1 and α_2

To get the cut of this region we will look at the vertices whose edges contribute to the cut. We will look at them in the following order: the first set of white vertices clockwise to the top black vertex connecting to the black and grey vertices, the other sets of white vertices connecting to the black and grey vertices, the diagonals connecting each black vertex to its opposite grey vertex, and the other black vertices connecting to the grey vertices.

Let's consider the first set of white vertices clockwise from the top black vertex. There are $\frac{P}{2}$ white vertices in this set. Also there is a combination of r black and grey vertices clockwise from this set of white vertices to the set of white vertices opposite of them and the remaining r black and grey vertices counter-clockwise from this set of white vertices to the set of white vertices opposite of them. As shown in Figure 2.13 there are three black and grey vertices clockwise and counter-clockwise from the first set of white

vertices clockwise from the top black vertex to the set of white vertices opposite of them.

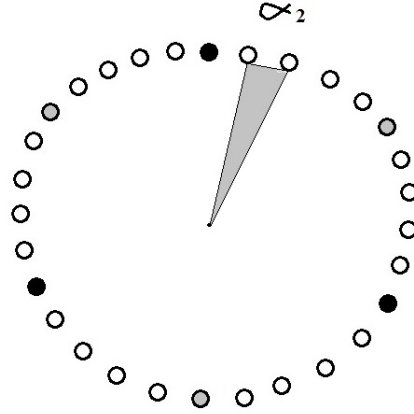


Figure 2.13: Region α_2 of $K_{3,3,24}$

Consider the first white vertex clockwise from the top black vertex. The combination of r black and grey vertices counter-clockwise to this white vertex to the white vertex opposite of it, will not contribute any edges to the cut of region α_2 . However, the remaining r black and grey vertices clockwise to this white vertex to the white vertex opposite it, will each contribute an edge to the cut of region α_2 .

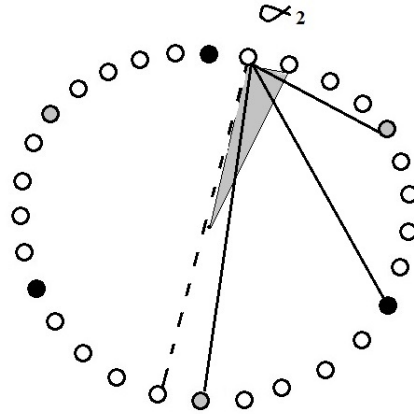


Figure 2.14: First Clockwise White Vertex Edge Contribution to α_2 of $K_{3,3,24}$

Consider the second white vertex clockwise from the top black vertex. The r

black and grey vertices clockwise to this white vertex to the white vertex opposite of it will not contribute any edges to the cut of region α_2 . However, r black and grey vertices counter-clockwise to the white vertex to the white vertex opposite it will each contribute an edge to the cut of region α_2 . This pattern will continue for each white vertex in the first set of white vertices clockwise to the top black vertex. Thus, this set of white vertices will contribute a total of $r \binom{p}{2}$ edges to the cut of the region α_2 .

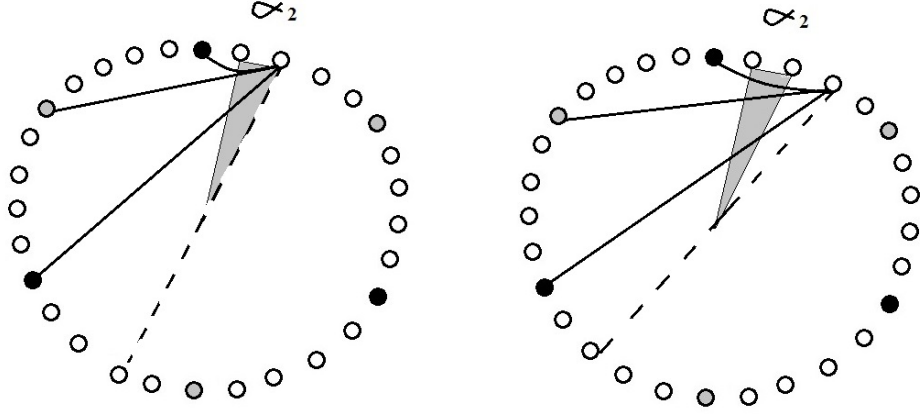


Figure 2.15: Second and Third Clockwise White Vertex Edge Contribution to α_2 of $K_{3,3,24}$

Now we need to consider the remaining white vertices. We are first going to look at each set of white vertices clockwise from the first set of white vertices we just considered and then look at the set of white vertices counter-clockwise to the same set. For each white vertex in the first set of white vertices clockwise to the set of white vertices of region α_2 , there is a combination of r black and grey vertices clockwise and counter-clockwise to this set of white vertices. The r black and grey vertices clockwise to the set of white vertices will contribute no edges to the cut of the region α_2 . So consider the remaining r black and grey vertices counter-clockwise to the set of white vertices being considered. There is now one grey vertex between the first and second set of white vertices that will not contribute an edge to the cut of the region α_2 . Thus, there will be $r - 1$ edges contributing to the cut of the region α_2 from each white vertex in this set, making a total contribution of $(r - 1) \binom{p}{2}$ edges contributing to the cut of the region α_2 .

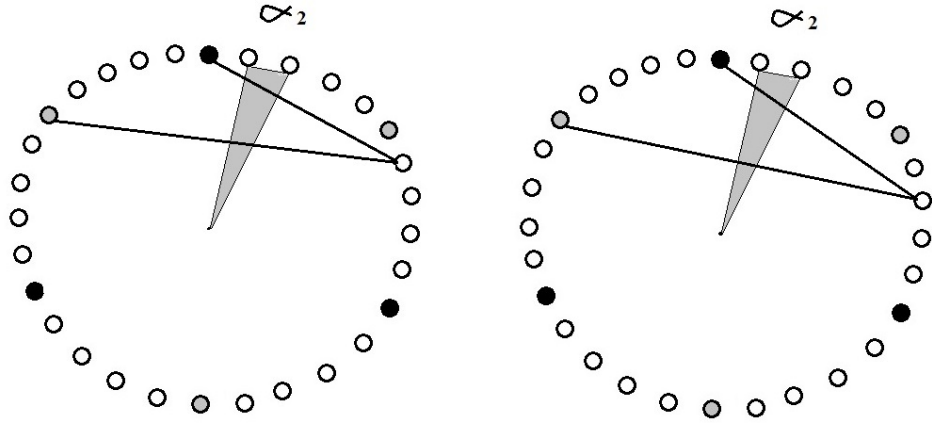


Figure 2.16: Second Set of Clockwise White Vertex Edge Contribution to α_2 of $K_{3,3,24}$

This same pattern happens for each set of white vertices clockwise from the first set. However, for each set you move clockwise, there is an additional grey or black vertex between the first set of white vertices and the set of white vertices being considered. For example, for the third set of white vertices clockwise from the top black vertex there will be two grey and black vertices between the first and third set of white vertices resulting in $r - 2$ edges contributing to the cut of the region of α_2 . This makes a total of $(r - 2) \binom{p}{2}$ edges contributing to the cut of the region α_2 from the third set of white vertices clockwise from the top black vertex. So the fourth set of white vertices will contribute a total of $(r - 3) \binom{p}{2}$ edges to the cut of the region α_2 . Continuing this pattern for each set of white vertices clockwise to the set of white vertices containing α_2 we will have the following edge contribution.

$$\begin{aligned}
& [\frac{p}{2}(r-1)] + [\frac{p}{2}(r-2)] + [\frac{p}{2}(r-3)] + \dots + [\frac{p}{2}(r-r)] \\
&= \sum_{n=1}^r \frac{p}{2}(r-n) \\
&= \frac{p}{2} \left[\sum_{n=1}^r r - \sum_{n=1}^r n \right] \\
&= \frac{p}{2} \left[r(r) - \frac{r(r+1)}{2} \right] \\
&= \frac{p(r^2-r)}{4}
\end{aligned}$$

Considering the other sets of white vertices counter-clockwise from the top black vertex the same pattern will occur contributing another $\frac{p(r^2-r)}{4}$ edges to the cut of region α_2 .

Now let's consider the diameters. Recall in this arrangement for every black vertex there will be a grey vertex directly across from it within the cyclic embedding. Also directly across from a white vertex there will always be another white vertex. Since white vertices cannot connect to white vertices there will only be r diameters connecting each black vertex to the one grey vertex directly across from it. Since the diameters cannot travel straight through the center of the cyclic embedding we will alternate each diameter clockwise then counter-clockwise around the center as shown in Figure 2.17. This will minimize the number of diameters crossing through each region. Therefore, either $\frac{r-1}{2}$ diameters or $\frac{r+1}{2}$ diameters will be contributing to the cut of each region. Since we are considering the maximum cut we will assume the cut of region α_2 contains $\frac{r+1}{2}$ diameters.

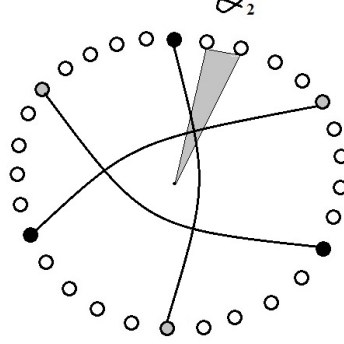


Figure 2.17: Diameters Contributing to The Cut of Region α_2 of $K_{3,3,24}$

The last set of edges that need to be considered are the non-diameter edges connecting black vertices to the grey vertices. So for each black vertex there are $\frac{r-1}{2}$ grey vertices clockwise and counter-clockwise to the black vertex being considered to the grey vertex directly across from it. The top black vertex will contribute $\frac{r-1}{2}$ edges to the cut of region α_2 by connecting to each $\frac{r-1}{2}$ grey vertices clockwise to the top black vertex. Now consider the black vertices clockwise from the top black vertex to the bottom grey vertex. The edges connecting to the grey vertices clockwise from each black vertex will not contribute any edges to the cut of region α_2 . So consider the grey vertices counterclockwise to each black vertex. The top black vertex and the black vertex being considered will have a multiple of grey vertices between them that will not contribute an edge to the cut of region α_2 . This number of grey vertices will increase by an increment of one for each black vertex that is farther from the top black vertex. Thus, the first black vertex clockwise from the top black vertex will contribute $\frac{r-1}{2} - 1$ edges, the second black vertex clockwise from the top black vertex will contribute $\frac{r-1}{2} - 2$ edges, and the third black vertex clockwise from the top black vertex will contribute $\frac{r-1}{2} - 3$ edges. This pattern will continue until the last black vertex clockwise from the top black vertex just before the bottom grey vertex is reached. Thus, the black vertices clockwise from the top black vertex will contribute the following edges to the cut of region α_2 .

$$\begin{aligned}
& \left[\frac{r-1}{2} - 1\right] + \left[\frac{r-1}{2} - 2\right] + \left[\frac{r-1}{2} - 3\right] + \dots + \left[\frac{r-1}{2} - \frac{r-1}{2}\right] \\
&= \sum_{n=1}^{\frac{r-1}{2}} \frac{r-1}{2} - n \\
&= \frac{1}{2} \left[\sum_{n=1}^{\frac{r-1}{2}} r - \sum_{n=1}^{\frac{r-1}{2}} 1 \right] - \sum_{n=1}^{\frac{r-1}{2}} n \\
&= \frac{1}{2} \left[r \left(\frac{r-1}{2} \right) - \frac{r-1}{2} \right] - \left(\frac{\frac{r-1}{2} \left(\frac{r-1}{2} + 1 \right)}{2} \right) \\
&= \frac{r^2 - 4r + 3}{8}
\end{aligned}$$

Considering the other black vertices counter-clockwise from the top black vertex the same pattern will occur contributing another $\frac{r^2 - 4r + 3}{8}$ edges to the cut of region α_2 .

Now that all possible edges contributing to the cut of region α_2 have been considered, the total edges contributing to the cut are as follows.

$$\begin{aligned}
cut(\alpha_2) &= r \left(\frac{p}{2} \right) + 2 \left(\frac{p(r^2 - r)}{4} \right) + \frac{r+1}{2} + \frac{r-1}{2} + 2 \left(\frac{r^2 - 4r + 3}{8} \right) \\
&= \frac{(2p+1)r^2 + 3}{4}
\end{aligned}$$

Now let's consider the other cuts within the set of one black vertex, one grey vertex, and p white vertices that will have a similar cut to the region α_2 . The cuts of the regions between two consecutive white vertices will be less than or equal to the cut of region α_2 . For each cut there is the same combination of r black and grey vertices clockwise each region to the opposite region and there is the same combination of r black and grey vertices counter-clockwise each region to the opposite region. Thus, each region will be symmetrical. Also, if we consider the white vertex counter-clockwise the region being

considered, the r edges from this white vertex that contributed to the cut of its adjacent region will no longer contribute to the cut of the region now being considered. This is due to this white vertex now being counter-clockwise to the region being considered and any edges connecting to the combination of r black and grey vertices also counter-clockwise to this region no longer contribute to the cut. However, the r edges from this white vertex that were not contributing to the cut of the adjacent region will now contribute to the cut of the region being considered for the same reason. Therefore, the r edges lost are regained resulting in an equivalent number of non-diameter edges contributing to the cut of each region similar to α_2 . Lastly, The edges from diameters will contribute the same number of edges or one edge less than the region adjacent to it.

Since the cut of all regions for this cyclic arrangement have been evaluated, we claim the upperbound for the linear cutwidth of $K_{r,r,pr}$ for r odd and p even is,

$$ccw(K_{r,r,pr}) \leq \frac{(2p+1)r^2+3}{4}$$

Now, since the upper bound and the lower bound are the same for p even, we claim the cyclic cutwidth of $K_{r,r,pr}$ for r odd and p even is,

$$ccw(K_{r,r,pr}) = \frac{(2p+1)r^2+3}{4}$$

So as you can see thus far, the lowerbound found for the cyclic cutwidth for complete tripartite graphs $K_{r,r,pr}$, for r odd and p a natural number, is relating agreeably with the upperbound. However, we are only halfway there. We now need to consider the final case, when p is odd. As you will see, case 2 is a little more extensive than case 1 since the arrangement of the cyclic embedding used throughout this proof is less symmetrical. With p being odd, each set of white vertices on either side of each grey vertex are no longer equivalent. This results in the set of white vertices counter-clockwise to the grey vertex containing one more white vertex than the set of white vertices clockwise to the grey vertex. So let's see how this effects the following proof.

2.2.2 Case 2: p is Odd

Let p be odd. Recall when p is odd the arrangement of the cyclic embedding for complete tripartite graph $K_{r,r,pr}$, for r odd, will consist of one black vertex followed

by $\frac{p+1}{2}$ white vertices, followed by a grey vertex, followed by $\frac{p-1}{2}$ white vertices and then the pattern repeats. With this case we need to consider four different regions where the maximum cut may lie: the region immediately clockwise to the black vertex, call it α_1 , the region immediately clockwise to the grey vertex, call it α_2 , the region immediately clockwise to the first white vertex of the set with $\frac{p+1}{2}$ white vertices, call it α_3 , and the region immediately clockwise to the first white vertex of the set with $\frac{p-1}{2}$ white vertices, call it α_4 . The remaining regions will be similar to either α_1 , α_2 , α_3 , or α_4 , and will be discussed further after an upperbound of the cyclic cutwidth is found for each region.

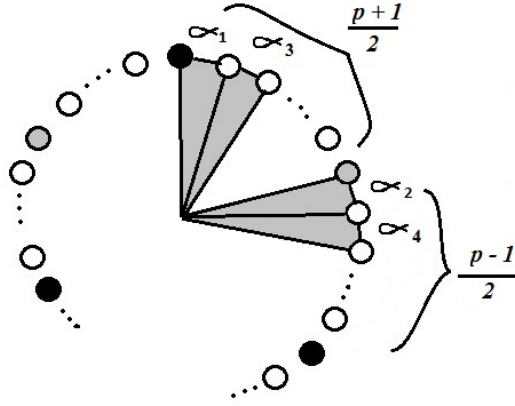


Figure 2.18: Regions α_1 , α_2 , α_3 , and α_4

Note that when p is odd there will be no diameters to consider. Also recall the non-diameter edges will take the shortest route within the cyclic embedding to connect two vertices. This will ensure each cut to be minimized so that not one or more regions are overloaded with edges.

Consider region α_1 . To get the cut of this region we will look at the vertices whose edges contribute to the cut. We will look at them in the following order: the top black vertex connecting to the grey and white vertices, the remaining black vertices connecting to the grey and white vertices, and the grey vertices connecting to the white vertices.

Let's first consider the edges connecting the top black vertex to the grey and

white vertices that will contribute to the cut of region α_1 . Since there is an odd number of vertices, there are no diameters and each vertex will be directly across from a region. Thus, there would be a combination of $\frac{pr+r}{2}$ white and grey vertices clockwise from the top black vertex to the opposite region and the remaining $\frac{pr+r}{2}$ white and grey vertices counter-clockwise from the top black vertex to the opposite region. The grey and white vertices counter-clockwise from the top black vertex to the opposite region will not contribute any edges to the cut of region α_1 . However, each grey and white vertex clockwise from the top black vertex to the opposite region will contribute an edge to the cut. Thus, the top black vertex will contribute a total of $\frac{pr+r}{2}$ edges to the cut of region α_1 . As you can see in Figure 2.19 only the grey and white vertices clockwise from the top black vertex to the opposite region will contribute an edge to the cut of region α_1 .

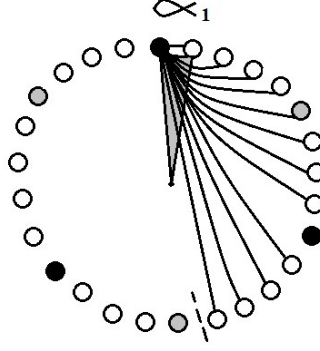


Figure 2.19: Top Black Vertex Edge Contribution to α_1 of $K_{3,3,21}$

We now need to consider the remaining $r - 1$ black vertices. There will be $\frac{r-1}{2}$ black vertices clockwise from the top black vertex to its opposite region and $\frac{r-1}{2}$ black vertices counter-clockwise from the top black vertex to its opposite region. Also, there will be a combination of $\frac{pr+r}{2}$ grey and white vertices clockwise from a black vertex to its opposite region and the remaining $\frac{pr+r}{2}$ grey and white vertices counter-clockwise from the same black vertex to its opposite region. Therefore, between each pair of consecutive black vertices there will be a set of $p + 1$ grey and white vertices. Thus, each black vertex clockwise and counter-clockwise from the top black vertex to its opposite region will have a multiple of $p + 1$ grey and white vertices between the top black vertex and the black vertex being considered. As you can see in Figure 2.20 there is one black vertex clockwise

and counter-clockwise from the top black vertex to its opposite region. There is a total of eight grey and white vertices between each black vertex, one grey vertex and seven white vertices.

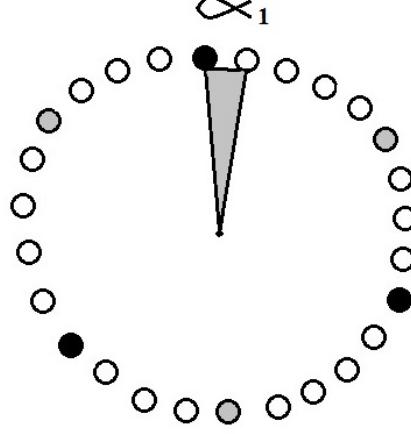


Figure 2.20: Remaining $r - 1$ Black Vertices Arrangement of $K_{3,3,21}$

Let's consider the first black vertex clockwise from the top black vertex. The grey and white vertices clockwise to this black vertex to its opposite region will not contribute any edges to the cut of region α_1 . So consider the remaining $\frac{pr+r}{2}$ grey and white vertices counter-clockwise. There will be $p+1$ grey and white vertices between this black vertex and the top black vertex that also will not contribute any edges to the cut of region α_1 . However, the remaining $\frac{pr+r}{2} - (p+1)$ grey and white vertices will each contribute an edge to the cut of region α_1 . See Figure 2.21 for an example.

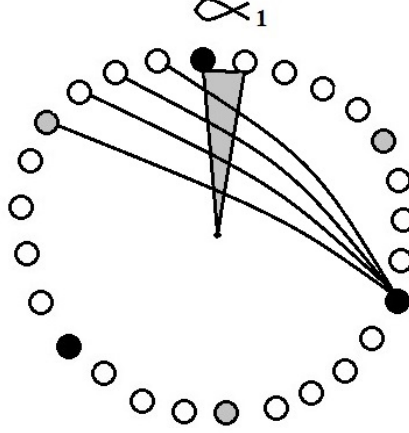


Figure 2.21: First Clockwise Black Vertex Edge Contribution to α_1 of $K_{3,3,21}$

Now consider the second black vertex clockwise from the top black vertex. Similarly, the grey and white vertices clockwise to this black vertex to its opposite region will not contribute any edges to the cut of region α_1 . Considering the remaining $\frac{pr+r}{2}$ grey and white vertices counter-clockwise, there will be $2(p+1)$ grey and white vertices between this black vertex and the top black vertex that will also not contribute any edges to the cut of region α_1 . Therefore, the remaining $\frac{pr+r}{2} - 2(p+1)$ grey and white vertices will each contribute an edge to the cut of region α_1 . This pattern will continue until we reach the region opposite the top black vertex. Thus, the total edges contributing to the cut of region α_1 from each $\frac{r-1}{2}$ black vertex clockwise from the top black vertex will be as follows. Note the calculations are similar to case 1, thus for analogous calculations following we will omit details.

$$\begin{aligned}
 \left[\frac{pr+r}{2} - (p+1) \right] + \left[\frac{pr+r}{2} - 2(p+1) \right] + \dots + \left[\frac{pr+r}{2} - \frac{r-1}{2}(p+1) \right] &= \sum_{n=1}^{\frac{r-1}{2}} \frac{(p+1)(r-2n)}{2} \\
 &= \frac{(p+1)(r-1)^2}{8}
 \end{aligned}$$

Considering the other $\frac{r-1}{2}$ black vertices counter-clockwise from the top black vertex, the same pattern will occur contributing another $\frac{(p+1)(r-1)^2}{8}$ edges to the cut of

region α_1 .

Now we need to consider the edges connecting the grey vertices to the white vertices that will contribute to the cut of region α_1 . There will be $\frac{r-1}{2}$ grey vertices clockwise from the top black vertex to its opposite region and $\frac{r+1}{2}$ grey vertices counter-clockwise from the top black vertex to its opposite region. Let's consider the grey vertices clockwise to the top black vertex first. For each grey vertex clockwise to the top black vertex to the region opposite it there are $\frac{pr-1}{2}$ white vertices clockwise to the grey vertex being considered that will not contribute an edge to the cut of region α_1 . However, there are $\frac{pr+1}{2}$ white vertices counter-clockwise to each of these grey vertices that need to be considered. So let's look at the first grey vertex clockwise to the top black vertex. There are $\frac{p+1}{2}$ white vertices between this grey vertex and the top black vertex that will not contribute an edge to the cut of region α_1 . However, the remaining $\frac{pr+1}{2} - \frac{p+1}{2}$ white vertices counter-clockwise to this grey vertex will each contribute an edge to the cut of region α_1 .

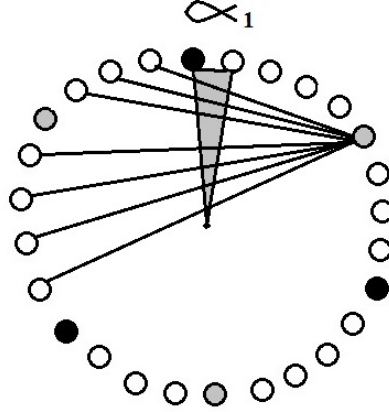


Figure 2.22: First Clockwise Grey Vertex Edge Contribution to α_1 of $K_{3,3,21}$

Now consider the second grey vertex clockwise to the top black vertex. Note between consecutive grey vertices there are p white vertices. Also between any two grey vertices there are a multiple of p white vertices. So considering the second grey vertex clockwise to the top black vertex, we still have the same $\frac{p+1}{2}$ white vertices between the top black vertex and the first grey vertex clockwise to the top black vertex that will not

contribute any edges to the cut of region α_1 . Also the p white vertices between the first and second grey vertices clockwise to the top black vertex will also not contribute any edges to the cut of region α_1 . However, the remaining $\frac{pr+1}{2} - \frac{p+1}{2} - p$ white vertices counter-clockwise to this grey vertex will each contribute an edge to the cut of region α_1 .

This pattern will continue until the last grey vertex just before the region opposite the top black vertex is reached. Thus, the total edges contributing to the cut of region α_1 from each $\frac{r-1}{2}$ grey vertex clockwise from the top black vertex will be as follows.

$$\begin{aligned} \left[\frac{pr+1}{2} - \frac{p+1}{2}\right] + \left[\frac{pr+1}{2} - \frac{p+1}{2} - p\right] + \dots + \left[\frac{pr+1}{2} - \frac{p+1}{2} - \left(\frac{r-1}{2} - 1\right)p\right] &= \sum_{n=1}^{\frac{r-1}{2}} \frac{p[(r-1)-2(n-1)]}{2} \\ &= \frac{p(r^2-1)}{8} \end{aligned}$$

Now consider the other $\frac{r+1}{2}$ grey vertices counter-clockwise from the top black vertex to the region opposite of it. Note, for each grey vertex counter-clockwise to the top black vertex to the region opposite it there are now $\frac{pr+1}{2}$ white vertices counter-clockwise to the grey vertex being considered that will not contribute any edges to the cut of region α_1 . However, there are $\frac{pr-1}{2}$ white vertices clockwise to each of these grey vertices that need to be considered. So let's look at the first grey vertex counter-clockwise to the top black vertex. There are now $\frac{p-1}{2}$ white vertices between this grey vertex and the top black vertex that will not contribute an edge to the cut of region α_1 . However, the remaining $\frac{pr-1}{2} - \frac{p-1}{2}$ white vertices clockwise to this grey vertex will each contribute an edge to the cut of region α_1 . Also between consecutive grey vertices there are still p white vertices, and between any two grey vertices there is still a multiple of p white vertices. So a pattern similar to the other grey vertices occurs. Thus, the total edges contributing to the cut of region α_1 from each $\frac{r+1}{2}$ grey vertex clockwise from the top black vertex will be as follows.

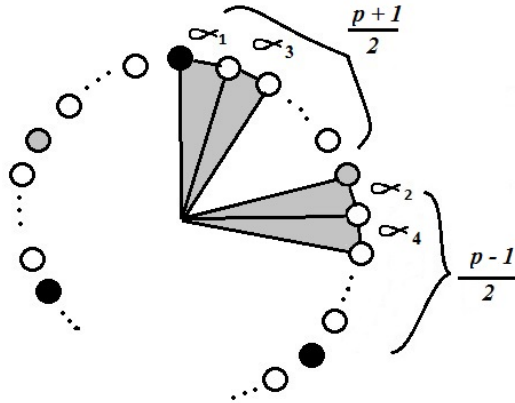
$$\begin{aligned}
\left[\frac{pr-1}{2} - \frac{p-1}{2}\right] + \left[\frac{pr-1}{2} - \frac{p-1}{2} - p\right] + \dots + \left[\frac{pr-1}{2} - \frac{p-1}{2} - \left(\frac{r-1}{2} - 1\right)p\right] &= \sum_{n=1}^{\frac{r-1}{2}} \frac{p[(r-1)-2(n-1)]}{2} \\
&= \frac{p(r^2-1)}{8}
\end{aligned}$$

Now that all possible edges contributing to the cut of region α_1 have been considered, the total edges contributing to the cut are as follows.

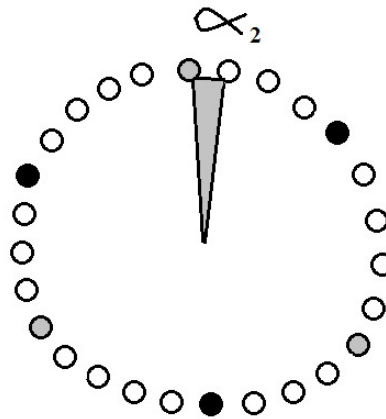
$$\begin{aligned}
cut(\alpha_1) &= \frac{r(p+1)-1}{2} + \frac{r+1}{2} + 2 \left[\frac{(p+1)(r-1)^2}{8} - \frac{r-1}{4} \right] + 2 \left[\frac{p(r+1)(r-1)}{8} \right] \\
&= \frac{(2p+1)r^2+1}{4}
\end{aligned}$$

Now let's consider the other cuts within the set of one black vertex, one grey vertex, and p white vertices that will have a similar cut to the region α_1 . The cut of the region immediately counter-clockwise to the grey vertex will be equal to the cut of region α_1 since the two regions are symmetrical.

Now that an upper bound has been found for the region α_1 and the regions similar to it, we now need to consider the remaining regions. So consider region α_2 .

Figure 2.23: Regions α_1 , α_2 , α_3 , and α_4

To get the cut of region α_2 we will look at the vertices whose edges contribute to the cut. To evaluate the cut of this region rotate the cyclic embedding counter-clockwise such that the grey vertex is at the top of the cyclic embedding. We will look at the edge contribution in the following order: the top grey vertex connecting to the black and white vertices, the remaining grey vertices connecting to the black and white vertices, and the black vertices connecting to the white vertices.

Figure 2.24: Rotation of Cyclic Embedding for α_2 of $K_{3,3,21}$

Let's first consider the edges connecting the top grey vertex to the black and

white vertices that will contribute to the cut of region α_2 . Since there is an odd number of vertices, there are no diameters and each vertex will be directly across from a region. Thus, there would be a combination of $\frac{pr+r}{2}$ white and black vertices clockwise from the top grey vertex to the opposite region and the remaining $\frac{pr+r}{2}$ white and black vertices counter-clockwise from the top grey vertex to the opposite region. The black and white vertices counter-clockwise from the top grey vertex to the opposite region will not contribute any edges to the cut of region α_2 . However, each black and white vertex clockwise from the top grey vertex to the opposite region will contribute an edge to the cut. Thus, the top grey vertex will contribute a total of $\frac{pr+r}{2}$ edges to the cut of region α_2 . As you can see in Figure 2.25 only the black and white vertices clockwise from the top grey vertex to the opposite region will contribute an edge to the cut of region α_2 .

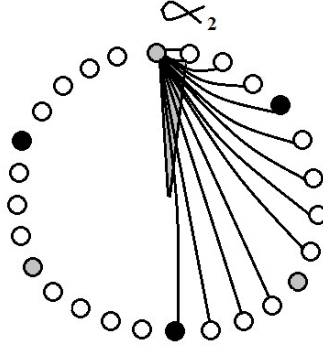


Figure 2.25: Top Grey Vertex Edge Contribution to α_2 of $K_{3,3,21}$

We now need to consider the remaining $r - 1$ grey vertices. There will be $\frac{r-1}{2}$ grey vertices clockwise from the top grey vertex to its opposite region and $\frac{r-1}{2}$ grey vertices counter-clockwise from the top grey vertex to its opposite region. Also, there will be a combination of $\frac{pr+r}{2}$ black and white vertices clockwise from a grey vertex to its opposite region and the remaining $\frac{pr+r}{2}$ black and white vertices counter-clockwise from the same grey vertex to its opposite region. Therefore, between each pair of consecutive grey vertices there will be a set of $p + 1$ black and white vertices. Thus, each grey vertex clockwise and counter-clockwise from the top grey vertex to its opposite region will have a multiple of $p + 1$ black and white vertices between the top grey vertex and the grey vertex being considered. As you can see in Figure 2.26 there is one grey vertex clockwise

and counter-clockwise from the top grey vertex to its opposite region. There is a total of eight black and white vertices between each grey vertex, one black vertex and seven white vertices.

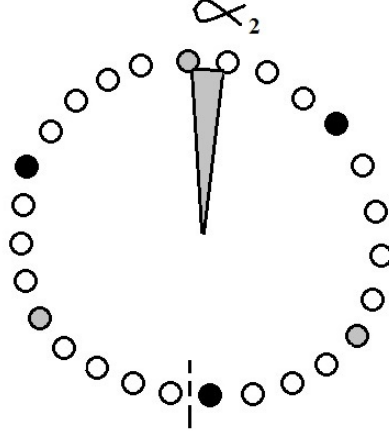


Figure 2.26: Remaining $r - 1$ Black Vertices Arrangement of $K_{3,3,21}$

Let's consider the first grey vertex clockwise from the top grey vertex. The black and white vertices clockwise to this grey vertex to its opposite region will not contribute any edges to the cut of region α_2 . So consider the remaining $\frac{pr+r}{2}$ black and white vertices counter-clockwise. There will be $p + 1$ black and white vertices between this grey vertex and the top grey vertex that also will not contribute any edges to the cut of region α_1 . However, the remaining $\frac{pr+r}{2} - (p + 1)$ black and white vertices will each contribute an edge to the cut of region α_1 . See Figure 2.27 for an example.

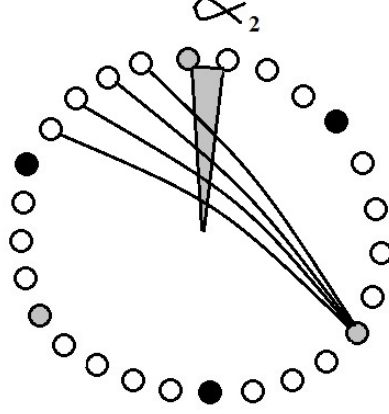


Figure 2.27: First Clockwise Grey Vertex Edge Contribution to α_2 of $K_{3,3,21}$

Now consider the second grey vertex clockwise from the top grey vertex. Similarly the black and white vertices clockwise to this grey vertex to its opposite region will not contribute any edges to the cut of region α_2 . Considering the remaining $\frac{pr+r}{2}$ black and white vertices counter-clockwise, there will be $2(p+1)$ black and white vertices between this grey vertex and the top grey vertex that will also not contribute any edges to the cut of region α_2 . Therefore, the remaining $\frac{pr+r}{2} - 2(p+1)$ black and white vertices will each contribute an edge to the cut of region α_2 . This pattern will continue until we reach the region opposite the top grey vertex. Thus, the total edges contributing to the cut of region α_2 from each $\frac{r-1}{2}$ grey vertex clockwise from the top grey vertex will be as follows.

$$\begin{aligned}
 \left[\frac{pr+r}{2} - (p+1)\right] + \left[\frac{pr+r}{2} - 2(p+1)\right] + \dots + \left[\frac{pr+r}{2} - \frac{r-1}{2}(p+1)\right] &= \sum_{n=1}^{\frac{r-1}{2}} \frac{(p+1)(r-2n)}{2} \\
 &= \frac{(p+1)(r-1)^2}{8}
 \end{aligned}$$

Considering the other $\frac{r-1}{2}$ grey vertices counter-clockwise from the top grey vertex, the same pattern will occur contributing another $\frac{(p+1)(r-1)^2}{8}$ edges to the cut of region α_2 .

Now we need to consider the edges connecting the black vertices to the white vertices that will contribute to the cut of region α_2 . There will be $\frac{r+1}{2}$ black vertices clockwise from the top grey vertex to its opposite region and $\frac{r-1}{2}$ black vertices counter-clockwise from the top grey vertex to its opposite region. Let's consider the black vertices clockwise to the top grey vertex first. For each black vertex clockwise to the top grey vertex to the region opposite it there are $\frac{pr+1}{2}$ white vertices clockwise to the black vertex being considered that will not contribute an edge to the cut of region α_2 . However, there are the remaining $\frac{pr-1}{2}$ white vertices counter-clockwise to each of these black vertices that need to be considered. So let's look at the first black vertex clockwise to the top grey vertex. There are $\frac{p-1}{2}$ white vertices between this black vertex and the top grey vertex that will not contribute an edge to the cut of region α_2 . However, the remaining $\frac{pr-1}{2} - \frac{p-1}{2}$ white vertices counter-clockwise to this black vertex will each contribute an edge to the cut of region α_2 .

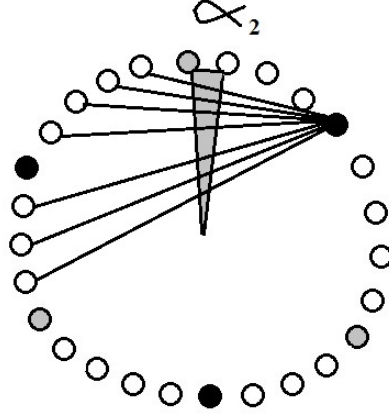


Figure 2.28: First Clockwise Black Vertex Edge Contribution to α_2 of $K_{3,3,21}$

Now consider the second black vertex clockwise to the top grey vertex. Note between consecutive black vertices there are p white vertices. Also between any two black vertices there is a multiple of p white vertices. So considering the second black vertex clockwise to the top grey vertex, we still have the same $\frac{p-1}{2}$ white vertices between the top grey vertex and the first black vertex clockwise to the top grey vertex that will not contribute any edges to the cut of region α_2 . Also the p white vertices between the first and second black vertices clockwise to the top grey vertex will also not contribute

any edges to the cut of region α_2 . However, the remaining $\frac{pr-1}{2} - \frac{p-1}{2} - p$ white vertices counter-clockwise to this black vertex will each contribute an edge to the cut of region α_2 .

This pattern will continue until the last black vertex just before the region opposite the top grey vertex is reached. Thus, the total edges contributing to the cut of region α_2 from each $\frac{r+1}{2}$ black vertex clockwise from the top black vertex will be as follows.

$$\begin{aligned} \left[\frac{pr-1}{2} - \frac{p-1}{2}\right] + \left[\frac{pr-1}{2} - \frac{p-1}{2} - p\right] + \dots + \left[\frac{pr-1}{2} - \frac{p-1}{2} - \left(\frac{r+1}{2} - 1\right)p\right] &= \sum_{n=1}^{\frac{r+1}{2}} \frac{p(r-2n+1)}{2} \\ &= \frac{p(r^2-1)}{8} \end{aligned}$$

Now consider the other $\frac{r-1}{2}$ black vertices counter-clockwise from the top grey vertex to the region opposite of it. Note, for each black vertex counter-clockwise to the top grey vertex to the region opposite it there are now $\frac{pr-1}{2}$ white vertices counter-clockwise to the black vertex being considered that will not contribute any edges to the cut of region α_2 . However, there are $\frac{pr+1}{2}$ white vertices clockwise to each of these black vertices that need to be considered. So let's look at the first black vertex counter-clockwise to the top grey vertex. There are now $\frac{p+1}{2}$ white vertices between this black vertex and the top grey vertex that will not contribute an edge to the cut of region α_2 . However, the remaining $\frac{pr+1}{2} - \frac{p+1}{2}$ white vertices clockwise to this black vertex will each contribute an edge to the cut of region α_2 . Also between consecutive black vertices there are still p white vertices, and between any two black vertices there is still a multiple of p white vertices. So a pattern similar to the other black vertices occurs. Thus, the total edges contributing to the cut of region α_2 from each $\frac{r-1}{2}$ black vertex clockwise from the top grey vertex will be as follows.

$$\begin{aligned} \left[\frac{pr+1}{2} - \frac{p+1}{2}\right] + \left[\frac{pr+1}{2} - \frac{p+1}{2} - p\right] + \dots + \left[\frac{pr+1}{2} - \frac{p+1}{2} - \left(\frac{r-1}{2} - 1\right)p\right] &= \sum_{n=1}^{\frac{r-1}{2}} \frac{p(r-2n+1)}{2} \\ &= \frac{p(r^2-1)}{8} \end{aligned}$$

Now that all possible edges contributing to the cut of region α_2 have been considered, the total edges contributing to the cut are as follows.

$$\begin{aligned} \text{cut}(\alpha_2) &= \frac{pr+r}{2} + 2 \left[\frac{(p+1)(r-1)^2}{8} \right] + \frac{p(r^2-1)}{8} + \frac{p(r^2-1)}{8} \\ &= \frac{(2p+1)r^2+1}{4} \end{aligned}$$

Now let's consider the other cuts within the set of one black vertex, one grey vertex, and p white vertices that will have a similar cut to the region α_2 . The cut of the region immediately counter-clockwise to the black vertex will be equal to the cut of region α_2 since the two regions are symmetrical.

Up to this point we have an upper bound for regions α_1 and α_2 , and we are right on track to completing case two. However, we still need to consider the remaining two regions α_3 and α_4 . Recall that α_3 and α_4 will address the regions within the white sets of vertices.

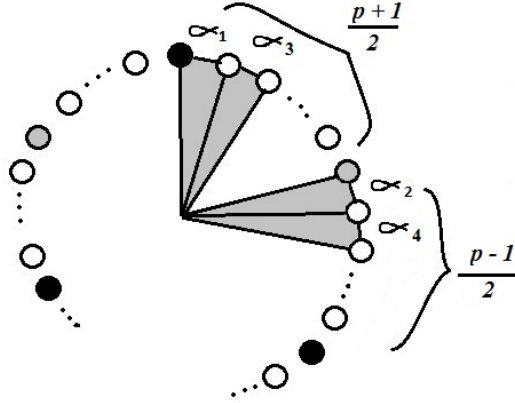


Figure 2.29: Regions $\alpha_1, \alpha_2, \alpha_3,$ and α_4

Let's first consider the cut of region α_3 . To get the cut of this region we will look at the vertices whose edges contribute to the cut. To evaluate the cut of this re-

gion, rotate the cyclic embedding counter-clockwise such that the most counter-clockwise white vertex of the set with $\frac{p+1}{2}$ white vertices is at the top of the cyclic embedding. We will look at the edge contribution in the following order: the top set of white vertices connecting to the black and grey vertices, the other sets of white vertices connecting to the black and grey vertices, and the black vertices connecting to the grey vertices.

So consider the top set of white vertices. There are $\frac{p+1}{2}$ white vertices in this set. Also there is a combination of r black and grey vertices clockwise from this set of white vertices to the set of white vertices opposite of them and the remaining r black and grey vertices counter-clockwise from this set of white vertices to the set of white vertices opposite of them. As shown in Figure 2.30 there is a combination of three black and grey vertices clockwise and counter-clockwise from the top set of white vertices to the set of white vertices opposite of them.

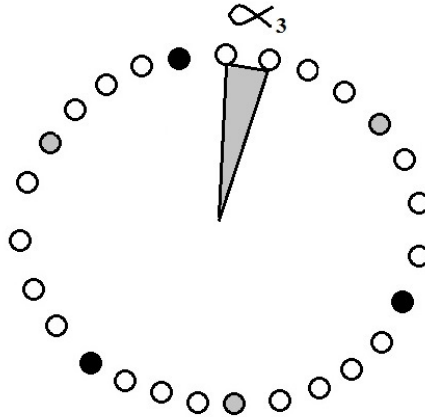


Figure 2.30: Region α_3 of $K_{3,3,21}$ Rotated

Consider the top white vertex of this set of white vertices. The combination of r black and grey vertices counter-clockwise to this white vertex to its opposite region, will not contribute any edges to the cut of region α_3 . However, the remaining r black and grey vertices clockwise to this white vertex to its opposite region, will each contribute an edge to the cut of region α_3 .

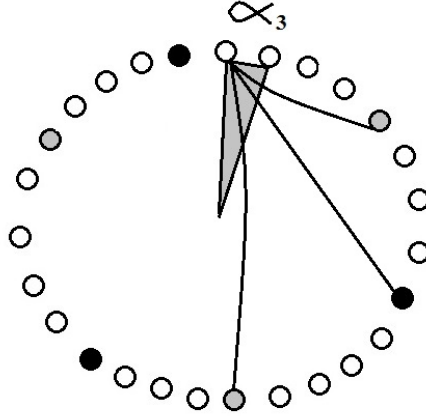


Figure 2.31: Top White Vertex Edge Contribution to α_3 of $K_{3,3,21}$

Now let's consider the first white vertex clockwise from the top white vertex. The r black and grey vertices clockwise to this white vertex to its opposite region will not contribute any edges to the cut of region α_3 . However, the r black and grey vertices counter-clockwise to this white vertex to its opposite region will each contribute an edge to the cut of region α_3 . This pattern will continue for each white vertex clockwise to the top white vertex within this set of white vertices. Thus, this set of white vertices will contribute a total of $r \left(\frac{p+1}{2} \right)$ edges to the cut of the region α_3 .

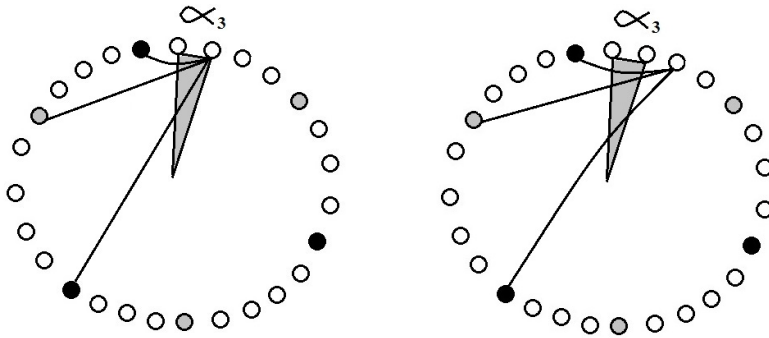


Figure 2.32: Top Set of White Vertices Edge Contribution to α_3 of $K_{3,3,21}$

Now we need to consider the remaining sets of white vertices. We are first going to look at each set of white vertices clockwise from the top set of white vertices we just considered, and then look at the set of white vertices counter-clockwise to the same set. For each white vertex in each set of white vertices clockwise to the top set, there is a combination of r black and grey vertices clockwise and counter-clockwise to the set of white vertices being considered. For each white vertex within each of these sets, the r black and grey vertices clockwise to each white vertex will contribute no edges to the cut of the region α_3 . So we need to consider the remaining r black and grey vertices counter-clockwise to each set of white vertices being considered. Recall that each set of consecutive white vertices alternate the number of vertices within the set. In this particular arrangement there are $\frac{p+1}{2}$ white vertices in the top set. Thus, the first set clockwise from the top set will have $\frac{p-1}{2}$ white vertices, the second set clockwise from the top set will have $\frac{p+1}{2}$ white vertices, and this pattern continues until we reach the white set of vertices opposite the top set. The bottom set of vertices will contain $\frac{p-1}{2}$ white vertices. So considering the first set of white vertices clockwise from the top set, there is now one grey vertex between the top set of white vertices and this set that will not contribute an edge to the cut of the region α_3 for each white vertex within this set. Thus, there will be $r - 1$ edges contributing to the cut of region α_3 from each white vertex in this set. Since this set contains $\frac{p-1}{2}$ white vertices, the total edge contribution to the cut of the region α_3 for this set is $(r - 1) \left(\frac{p-1}{2} \right)$.

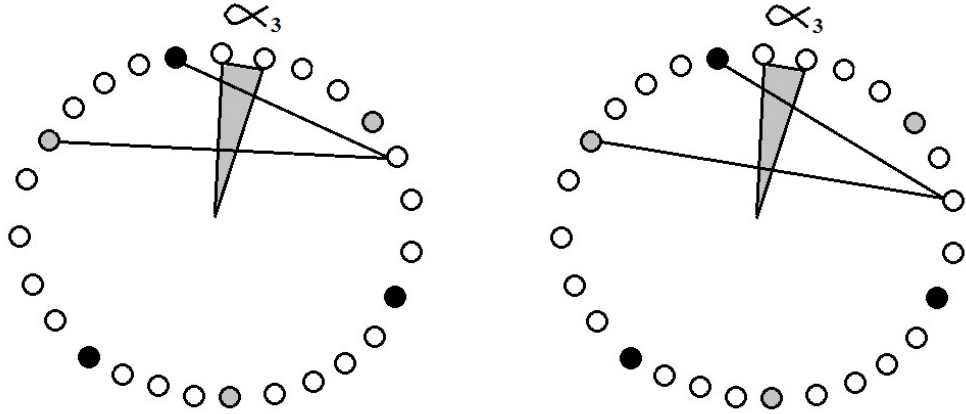


Figure 2.33: First Set of White Vertices Clockwise Top Set Edge Contribution to α_3 of $K_{3,3,21}$

This same pattern happens for each set of white vertices clockwise from the top set. However, for each set you move clockwise, there is an additional grey or black vertex between the top set of white vertices and the set of white vertices being considered. Also, the number of white vertices within each set is alternating. Thus, the second set of white vertices clockwise from the top set will contribute a total of $(r-2) \left(\frac{p+1}{2}\right)$ edges to the cut of region α_3 , the third set of white vertices clockwise from the top set will contribute a total of $(r-3) \left(\frac{p-1}{2}\right)$ edges to the cut of region α_3 , so on and so forth. Continuing this pattern for each set of white vertices clockwise to the top set of white vertices, we will have the following edge contribution. Note that we need to sum the edge contribution from the sets with $\frac{p+1}{2}$ white vertices separate from the sets with $\frac{p-1}{2}$ white vertices. So the following calculations are the sum of the edge contribution from the set with $\frac{p+1}{2}$ white vertices.

$$\begin{aligned}
& [(\frac{p+1}{2})(r-2)] + [(\frac{p+1}{2})(r-4)] + [(\frac{p+1}{2})(r-6)] + \dots + [(\frac{p+1}{2})(r-2(\frac{r-1}{2}))] \\
&= \sum_{n=1}^{\frac{r-1}{2}} (\frac{p+1}{2})(r-2n) \\
&= \frac{(p+1)(r^2-2r+1)}{8}
\end{aligned}$$

Now the following calculations are the sum of the edge contribution from the set with $\frac{p-1}{2}$ white vertices.

$$\begin{aligned}
& [(\frac{p-1}{2})(r-1)] + [(\frac{p-1}{2})(r-3)] + [(\frac{p-1}{2})(r-5)] + \dots + [(\frac{p-1}{2})(r-(2(\frac{r-1}{2})-1))] \\
&= \sum_{n=1}^{\frac{r-1}{2}} (\frac{p-1}{2})(r-(2n-1)) \\
&= \frac{(p-1)(r^2-1)}{8}
\end{aligned}$$

So combining the two sums, the total number of edges contributing to the cut of region α_3 from the sets of white vertices clockwise to the top set of white vertices

is $\frac{pr^2-pr-r+1}{4}$. Considering the other sets of white vertices counter-clockwise from the top set of white vertices, the pattern is the same. Thus, another $\frac{pr^2-pr-r+1}{4}$ edges will contribute to the cut of region α_3 .

The last set of edges that need to be considered are the edges connecting black vertices to the grey vertices. First consider the black vertex closest to the set of vertices opposite the top white set. The grey vertices counterclockwise to this black vertex will contribute no edges to the cut of region α_3 . However, the grey vertices clockwise from this black vertex also will not contribute any edges to the cut of region α_3 . This is due to there being no grey vertices between region α_3 and the region opposite this black vertex. Thus, this black vertex will not contribute any edges to the cut of region α_3 .

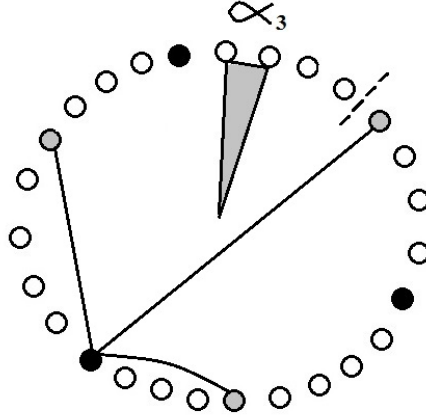


Figure 2.34: Black Vertex in Set Opposite Top White Set Edge Contribution to α_3 of $K_{3,3,21}$

So there will be $\frac{r-1}{2}$ black vertices clockwise from the top white vertex to its opposite region and $\frac{r-1}{2}$ black vertices counter-clockwise from the top white vertex to its opposite region that need to be considered. Let's consider the black vertices clockwise to the top white vertex first. For each black vertex clockwise to the top white vertex to the region opposite of the black vertex, there are $\frac{r-1}{2}$ grey vertices clockwise to the black vertex being considered that will not contribute an edge to the cut of region α_3 . However, there are the remaining $\frac{r+1}{2}$ grey vertices counter-clockwise to each of these black vertices that need to be considered. So let's look at the first black vertex clockwise to the top white vertex. There is one grey vertex between this black vertex and the top white vertex

that will not contribute an edge to the cut of region α_3 . However, the remaining $\frac{r+1}{2} - 1$ grey vertices counter-clockwise to this black vertex will each contribute an edge to the cut of region α_3 .

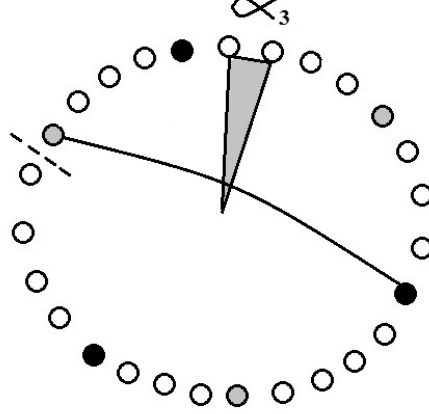


Figure 2.35: First Black Vertex Clockwise Top White Vertex Edge Contribution to α_3 of $K_{3,3,21}$

Now consider the second black vertex clockwise to the top white vertex. There are now two grey vertices between this black vertex and the top white vertex that will not contribute an edge to the cut of region α_3 . However, the remaining $\frac{r+1}{2} - 2$ grey vertices counter-clockwise to this black vertex will each contribute an edge to the cut of region α_3 . This pattern will continue until the last black vertex just before the region opposite the top white vertex is reached. Thus, the total edges contributing to the cut of region α_3 from each $\frac{r-1}{2}$ black vertex clockwise from the top white vertex will be as follows.

$$\begin{aligned} \left[\frac{r+1}{2} - 1\right] + \left[\frac{r+1}{2} - 2\right] + \dots + \left[\frac{r+1}{2} - \frac{r-1}{2}\right] &= \sum_{n=1}^{\frac{r-1}{2}} \left[\frac{r+1}{2} - n\right] \\ &= \frac{r^2-1}{8} \end{aligned}$$

Considering the other black vertices counter-clockwise from the top white vertex, a similar pattern will occur. However, there are a few slight changes. For each $\frac{r-1}{2}$ black vertices counter-clockwise from the top white vertex to its opposite region there are now

$\frac{r+1}{2}$ grey vertices counter-clockwise to each black vertex that will not contribute an edge to the cut of region α_3 . Also there are now $\frac{r-1}{2}$ grey vertices clockwise to each black vertex that need to be considered. The first black vertex counter-clockwise to the top white vertex will have no grey vertices between itself and the top white vertex. Thus, it will contribute $\frac{r-1}{2}$ edges to the cut of region α_3 .

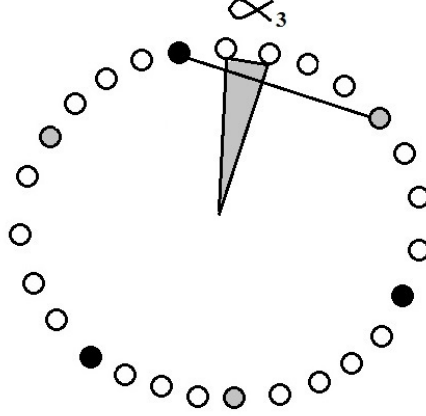


Figure 2.36: First Black Vertex Counter-Clockwise Top White Vertex Edge Contribution to α_3 of $K_{3,3,21}$

The second black vertex counter-clockwise to the top white vertex now has one grey vertex between this black vertex and the top white vertex that will not contribute an edge to the cut of region α_3 . However, the remaining $\frac{r-1}{2} - 1$ grey vertices clockwise to this black vertex will each contribute an edge to the cut of region α_3 . The third black vertex counter-clockwise to the top white vertex now has two grey vertices between this black vertex and the top white vertex that will not contribute an edge to the cut of region α_3 . However, the remaining $\frac{r-1}{2} - 2$ grey vertices clockwise to this black vertex will each contribute an edge to the cut of region α_3 . This pattern will continue until the last black vertex just before the region opposite the top white vertex is reached. Thus, the total edges contributing to the cut of region α_3 from each $\frac{r-1}{2}$ black vertex counter-clockwise from the top white vertex will be as follows.

$$\begin{aligned}
\left[\frac{r-1}{2}\right] + \left[\frac{r-1}{2} - 1\right] + \left[\frac{r-1}{2} - 2\right] + \dots + \left[\frac{r-1}{2} - \left(\frac{r-1}{2} - 1\right)\right] &= \sum_{n=1}^{\frac{r-1}{2}} \left[\frac{r-1}{2} - (n-1)\right] \\
&= \frac{r^2-1}{8}
\end{aligned}$$

Now that all possible edges contributing to the cut of region α_3 have been considered, the total edges contributing to the cut are as follows.

$$\begin{aligned}
cut(\alpha_3) &= r \left(\frac{p+1}{2}\right) + 2 \left(\frac{pr^2-pr-r+1}{4}\right) + 2 \left(\frac{r^2-1}{8}\right) \\
&= \frac{(2p+1)r^2+1}{4}
\end{aligned}$$

Now let's consider the other cuts within the set of one black vertex, one grey vertex, and p white vertices that will have a similar cut to the region α_3 . The cuts of the regions between two consecutive white vertices within the sets that contain $\frac{p+1}{2}$ white vertices, will be equal to the cut of region α_3 . For each cut there is the same combination of r black and grey vertices clockwise to each region to its opposite region, and there is the same combination of r black and grey vertices counter-clockwise to each region to its opposite region. Thus, each region will be symmetrical. Also, if we consider the white vertex counter-clockwise from the region being considered, the r edges from this white vertex that contributed to the cut of its adjacent region will no longer contribute to the cut of the region now being considered. This is due to this white vertex now being counter-clockwise to the region being considered and any edges connecting to the combination of r black and grey vertices also counter-clockwise to this region no longer contribute to the cut. However, the r edges from this white vertex that were not contributing to the cut of the adjacent region will now contribute to the cut of the region being considered for the same reason. Therefore, the r edges lost are regained resulting in an equivalent number edges contributing to the cut of each region similar to α_3 .

Lastly, we have region α_4 to evaluate. To get the cut of this region we will look at the vertices whose edges contribute to the cut. To evaluate the cut of this re-

gion, rotate the cyclic embedding counter-clockwise such that the most counter-clockwise white vertex of the set with $\frac{p-1}{2}$ white vertices is at the top of the cyclic embedding. We will look at the edge contribution in the following order: the top set of white vertices connecting to the black and grey vertices, the other sets of white vertices connecting to the black and grey vertices, and the black vertices connecting to the grey vertices.

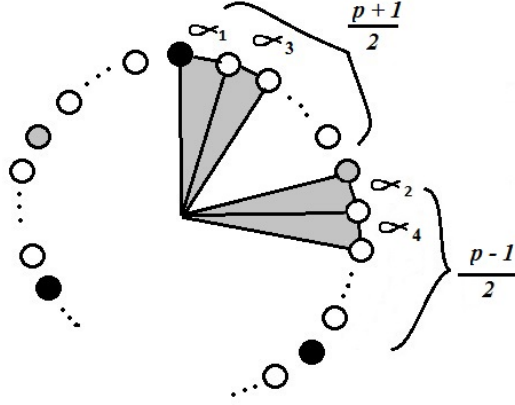
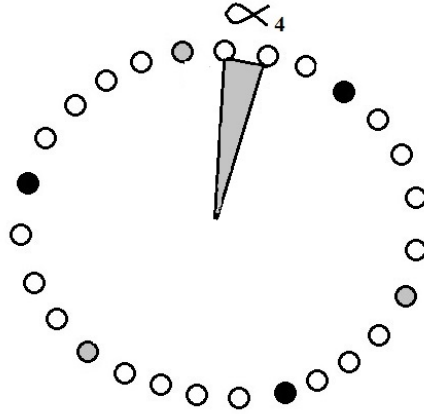
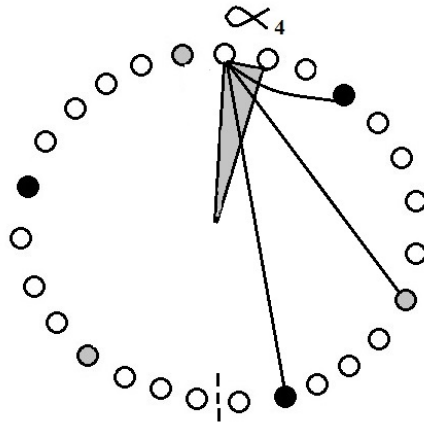


Figure 2.37: Regions α_1 , α_2 , α_3 , and α_4

So consider the top set of white vertices. There are $\frac{p-1}{2}$ white vertices in this set. Also there is a combination of r black and grey vertices clockwise from this set of white vertices to the set of white vertices opposite of them and the remaining r black and grey vertices counter-clockwise from this set of white vertices to the set of white vertices opposite of them. As shown in Figure 2.38 there is a combination of three black and grey vertices clockwise and counter-clockwise from the top set of white vertices to the set of white vertices opposite of them.

Figure 2.38: Region α_4 of $K_{3,3,21}$ Rotated

Consider the top white vertex of this set of white vertices. The combination of r black and grey vertices counter-clockwise to this white vertex to its opposite region, will not contribute any edges to the cut of region α_4 . However, the remaining r black and grey vertices clockwise to this white vertex to its opposite region, will each contribute an edge to the cut of region α_4 .

Figure 2.39: Top White Vertex Edge Contribution to α_4 of $K_{3,3,21}$

Now let's consider the first white vertex clockwise from the top white vertex. The r black and grey vertices clockwise to this white vertex to its opposite region will not contribute any edges to the cut of region α_4 . However, the r black and grey vertices

counter-clockwise to this white vertex to its opposite region will each contribute an edge to the cut of region α_4 . This pattern will continue for each white vertex clockwise to the top white vertex within this set of white vertices. Thus, this set of white vertices will contribute a total of $r \left(\frac{p-1}{2} \right)$ edges to the cut of the region α_4 .

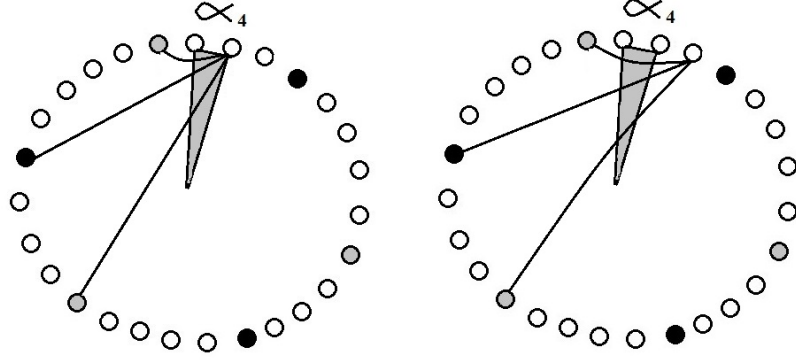


Figure 2.40: Top Set of White Vertices Edge Contribution to α_3 of $K_{3,3,21}$

Now we need to consider the remaining sets of white vertices. We are first going to look at each set of white vertices clockwise from the top set of white vertices we just considered, and then look at the set of white vertices counter-clockwise to the same set. For each white vertex in each set of white vertices clockwise to the top set, there is a combination of r black and grey vertices clockwise and counter-clockwise to this set of white vertices being considered. For each white vertex within each of these sets, the r black and grey vertices clockwise to each white vertex will contribute no edges to the cut of the region α_4 . So we need to consider the remaining r black and grey vertices counter-clockwise to each set of white vertices being considered. Recall that each set of consecutive white vertices alternate the number of vertices within the set. In this particular arrangement there are $\frac{p-1}{2}$ white vertices in the top set. Thus, the first set clockwise from the top set will have $\frac{p+1}{2}$ white vertices, the second set clockwise from the top set will have $\frac{p-1}{2}$ white vertices, and this pattern continues until we reach the white set of vertices opposite the top set. The bottom set of vertices will contain $\frac{p+1}{2}$ white vertices. So considering the first set of white vertices clockwise from the top set, there is now a black vertex between the top set of white vertices and this set, that will not contribute an edge to the cut of the region α_4 for each white vertex within this set. Thus,

there will be $r - 1$ edges contributing to the cut of region α_4 from each white vertex in this set. Since this set contains $\frac{p+1}{2}$ white vertices, the total edge contribution to the cut of the region α_4 for this set is $(r - 1) \left(\frac{p+1}{2} \right)$.

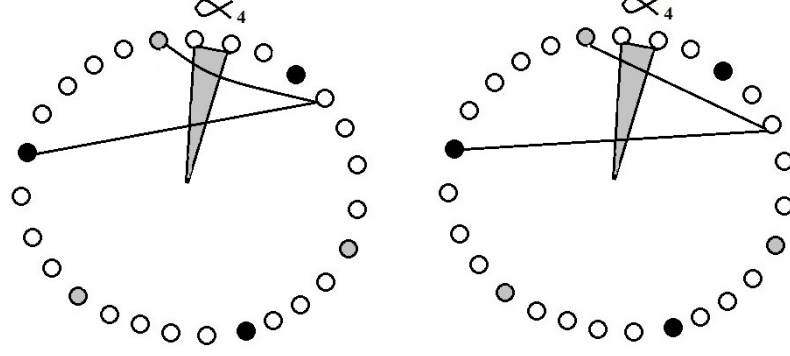


Figure 2.41: First Set of White Vertices Clockwise Top Set Edge Contribution to α_4 of $K_{3,3,21}$

This same pattern happens for each set of white vertices clockwise from the top set. However, for each set you move clockwise, there is an additional grey or black vertex between the top set of white vertices and the set of white vertices being considered. Also the number of white vertices within each set is alternating. Thus, the second set of white vertices clockwise from the top set will contribute a total of $(r - 2) \left(\frac{p-1}{2} \right)$ edges to the cut of region α_4 , the third set of white vertices clockwise from the top set will contribute a total of $(r - 3) \left(\frac{p+1}{2} \right)$ edges to the cut of region α_4 , so on and so forth. Continuing this pattern for each set of white vertices clockwise to the top set of white vertices, we will have the following edge contribution. Note that we need to sum the edge contribution from the sets with $\frac{p+1}{2}$ white vertices separate from the sets with $\frac{p-1}{2}$ white vertices. So the following calculations are the sum of the edge contribution from the set with $\frac{p-1}{2}$ white vertices.

$$\begin{aligned}
& [(\frac{p-1}{2})(r-2)] + [(\frac{p-1}{2})(r-4)] + [(\frac{p-1}{2})(r-6)] + \dots + [(\frac{p-1}{2})(r-2(\frac{r-1}{2}))] \\
&= \sum_{n=1}^{\frac{r-1}{2}} (\frac{p-1}{2})(r-2n) \\
&= \frac{(p-1)(r^2-2r+1)}{8}
\end{aligned}$$

Now the following calculations are the sum of the edge contribution from the set with $\frac{p+1}{2}$ white vertices.

$$\begin{aligned}
& [(\frac{p+1}{2})(r-1)] + [(\frac{p+1}{2})(r-3)] + [(\frac{p+1}{2})(r-5)] + \dots + [(\frac{p+1}{2})(r-(2(\frac{r-1}{2})-1))] \\
&= \sum_{n=1}^{\frac{r-1}{2}} (\frac{p+1}{2})(r-(2n-1)) \\
&= \frac{(p+1)(r^2-1)}{8}
\end{aligned}$$

So combining the two sums, the total number of edges contributing to the cut of region α_4 from the sets of white vertices clockwise from the top set of white vertices is $\frac{pr^2-pr+r-1}{4}$. Considering the other sets of white vertices counter-clockwise from the top set of white vertices, the pattern is the same. Thus, another $\frac{pr^2-pr+r-1}{4}$ edges will contribute to the cut of region α_4 .

The last set of edges that need to be considered are the edges connecting black vertices to the grey vertices. There will be $\frac{r+1}{2}$ black vertices clockwise from the top white vertex to its opposite region and $\frac{r-1}{2}$ black vertices counter-clockwise from the top white vertex to its opposite region that need to be considered. Let's consider the black vertices clockwise to the top white vertex first. For each black vertex clockwise to the top white vertex to the region opposite of the black vertex, there are $\frac{r-1}{2}$ grey vertices clockwise to the black vertex being considered that will not contribute an edge to the cut of region α_4 . However, there are the remaining $\frac{r+1}{2}$ grey vertices counter-clockwise

to each of these black vertices that need to be considered. So let's look at the first black vertex clockwise to the top white vertex. There will be no grey vertices between itself and the top white vertex thus it will contribute $\frac{r+1}{2}$ edges to the cut of region α_4 .

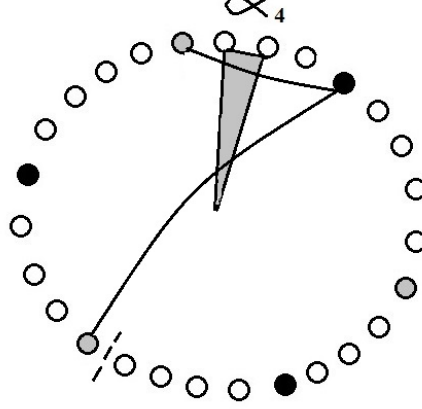


Figure 2.42: First Black Vertex Clockwise Top White Vertex Edge Contribution to α_4 of $K_{3,3,21}$

Now consider the second black vertex clockwise to the top white vertex. There is now one grey vertex between this black vertex and the top white vertex, that will not contribute an edge to the cut of region α_4 . However, the remaining $\frac{r+1}{2} - 1$ grey vertices counter-clockwise to this black vertex will each contribute an edge to the cut of region α_4 . This pattern will continue until the last black vertex just before the region opposite the top white vertex is reached. Thus, the total edges contributing to the cut of region α_4 from each $\frac{r+1}{2}$ black vertex clockwise from the top white vertex will be as follows.

$$\begin{aligned}
 \left\lceil \frac{r+1}{2} \right\rceil + \left\lceil \frac{r+1}{2} - 1 \right\rceil + \dots + \left\lceil \frac{r+1}{2} - \left(\frac{r-1}{2} - 1 \right) \right\rceil &= \sum_{n=1}^{\frac{r+1}{2}} \left[\frac{r+1}{2} - (n-1) \right] \\
 &= \frac{r^2+4r+3}{8}
 \end{aligned}$$

Considering the other black vertices counter-clockwise from the top white vertex, a similar pattern will occur. However, there are a few slight changes. For each $\frac{r-1}{2}$ black vertices counter-clockwise from the top white vertex to its opposite region, there are now

$\frac{r+1}{2}$ grey vertices counter-clockwise to each black vertex that will not contribute an edge to the cut of region α_4 . So now there are $\frac{r-1}{2}$ grey vertices clockwise to each black vertex that need to be considered. Also, the first black vertex counter-clockwise from the top white vertex now has one grey vertex between this black vertex and the top white vertex that will not contribute an edge to the cut of region α_4 . However, the remaining $\frac{r-1}{2} - 1$ grey vertices clockwise to this black vertex will each contribute an edge to the cut of region α_4 .

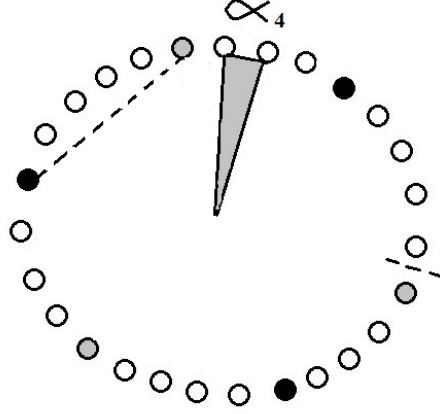


Figure 2.43: First Black Vertex Counter-Clockwise Top White Vertex Edge Contribution to α_4 of $K_{3,3,21}$

The second black vertex counter-clockwise to the top white vertex now has two grey vertices between this black vertex and the top white vertex that will not contribute an edge to the cut of region α_4 . However, the remaining $\frac{r-1}{2} - 2$ grey vertices clockwise to this black vertex will each contribute an edge to the cut of region α_4 . The third black vertex counter-clockwise to the top white vertex now has three grey vertices between this black vertex and the top white vertex that will not contribute an edge to the cut of region α_4 . However, the remaining $\frac{r-1}{2} - 3$ grey vertices clockwise to this black vertex will each contribute an edge to the cut of region α_4 . This pattern will continue until the last black vertex just before the region opposite the top white vertex is reached. Thus, the total edges contributing to the cut of region α_4 from each $\frac{r-1}{2}$ black vertex counter-clockwise from the top white vertex will be as follows.

$$\begin{aligned}
\left[\frac{r-1}{2} - 1\right] + \left[\frac{r-1}{2} - 2\right] + \left[\frac{r-1}{2} - 3\right] + \dots + \left[\frac{r-1}{2} - \frac{r-1}{2}\right] &= \sum_{n=1}^{\frac{r-1}{2}} \left[\frac{r-1}{2} - n\right] \\
&= \frac{r^2 - 4r + 3}{8}
\end{aligned}$$

Now that all possible edges contributing to the cut of region α_4 have been considered, the total edges contributing to the cut are as follows.

$$\begin{aligned}
cut(\alpha_4) &= r \left(\frac{p-1}{2}\right) + 2 \left(\frac{pr^2 - pr + r - 1}{4}\right) + \frac{r^2 + 4r + 3}{8} + \frac{r^2 - 4r + 3}{8} \\
&= \frac{(2p+1)r^2 + 1}{4}
\end{aligned}$$

Now let's consider the other cuts within the set of one black vertex, one grey vertex, and p white vertices that will have a similar cut to the region α_4 . The cuts of the regions between two consecutive white vertices, within the sets that contain $\frac{p-1}{2}$ white vertices, will be equal to the cut of region α_4 . For each cut there is the same combination of r black and grey vertices clockwise to each region to the opposite region and there is the same combination of r black and grey vertices counter-clockwise to each region to the opposite region. Thus, each region will be symmetrical. Also, if we consider the white vertex counter-clockwise from the region being considered, the r edges from this white vertex that contributed to the cut of its adjacent region will no longer contribute to the cut of the region now being considered. This is due to this white vertex now being counter-clockwise to the region being considered and any edges connecting to the combination of r black and grey vertices also counter-clockwise to this region no longer contribute to the cut. However, the r edges from this white vertex that were not contributing to the cut of the adjacent region will now contribute to the cut of the region being considered for the same reason. Therefore, the r edges lost are regained resulting in an equivalent number edges contributing to the cut of each region similar to α_4 .

Since the cut of all regions for this cyclic arrangement have been evaluated, we claim the upperbound for the linear cutwidth of $K_{r,r,pr}$ for r odd and p odd is,

$$ccw(K_{r,r,pr}) \leq \frac{(2p+1)r^2+1}{4}$$

Also, since the upper bound and the lower bound are the same for p odd, we claim the cyclic cutwidth of $K_{r,r,pr}$ for r odd and p odd is,

$$ccw(K_{r,r,pr}) = \frac{(2p+1)r^2+1}{4}$$

We now have claimed that the lowerbounds and upperbounds for the linear cutwidth of complete tripartite graph $K_{r,r,pr}$ for r odd and both p even and p odd are the same. Therefore, for a complete tripartite graph $K_{r,r,pr}$ where r is odd and p a natural number, the cyclic cutwidth is,

$$ccw(K_{r,r,pr}) = \begin{cases} \frac{(2p+1)r^2+1}{4} & \text{for } p \text{ odd} \\ \frac{(2p+1)r^2+3}{4} & \text{for } p \text{ even} \end{cases}$$

□

Chapter 3

Conclusion

Up to this point we have looked at the known different cases for the linear cutwidth and cyclic cutwidth of complete bipartite and tripartite graphs. In addition, we have looked at the link between the cyclic cutwidth and linear cutwidth of any graph. Further discoveries have been made for a new case of the cyclic cutwidth for the complete tripartite graph $K_{r,r,pr}$ where r is odd and p is a natural number. In this chapter we will discuss the known discoveries that have been explored for the complete n -partite graphs and the opportunities for new exploration.

3.1 Further Exploration of The Cyclic Cutwidth of Complete Bipartite Graphs

Even though many cases of the cyclic cutwidth of complete bipartite graphs have been proven, there are still other cases to be considered and they can be very tedious. Recall, a complete bipartite graph consists of two sets of vertices, V_1 and V_2 . The number of vertices within each set are as follows, $|V_1|=m$ and $|V_2|=n$. Currently there are algorithms to find the cyclic cutwidth of complete bipartite graphs when $m \leq n$ for m and n both even, $m = n$ for m and n both odd, and many cases where there are multiple restrictions on m and n for multiple combinations of m and n even and odd. These different cases start to address specific situations when $m \leq n$ for m and n both odd or one odd and the other even. For example $ccw(K_{m,n}) = \frac{mn+2}{4}$ when $m \equiv 2(\text{mod } 4)$, n odd, $2n \geq m$. So, currently we have the cyclic cutwidth for $m \leq n$ for m and n both even, but

are lacking in the category for $m \leq n$ for m and n both odd, or $m \leq n$ for one even and the other odd. Thus, we can consider further exploration to expand on different cases for $m \leq n$ for m and n both odd, or $m \leq n$ for one even and the other odd. Also we could look at similarities within the current cases to see if a summarized algorithm can be discovered.

3.2 Further Exploration of The Cyclic Cutwidth of Complete Tripartite Graphs

A similar situation arises for the cyclic cutwidth of complete tripartite graphs. Recall a complete tripartite graph consists of three sets of vertices, V_1 , V_2 , and V_3 . The number of vertices within each set are as follows, $|V_1|=r$, $|V_2|=s$, $|V_3|=t$. Currently there are algorithms to find the cyclic cutwidth of complete tripartite graphs when $r \leq s \leq t$ for r , s , and t all even, $r = s = t$ for r , s , and t all odd, and $r = s$ and $t = pr$ for r , s , and t all odd and p a natural number. However, we are still lacking in cases for $r \leq s \leq t$ for r , s , and t all odd, and $r \leq s \leq t$ for a combination of r , s , and t even or odd. Thus, further exploration to expand the different cases for $r \leq s \leq t$ for r , s , and t all odd, and $r \leq s \leq t$ for a combination of r , s , and t even or odd can be explored. The next step could be to consider the graph $K_{r,pr,q}$ where r is odd and p and q are natural numbers. As well, we could look at similarities within the current cases to see if a summarized algorithm can be discovered.

3.3 Further Exploration of The Cyclic Cutwidth of Complete n -Partite Graphs

The final step for the cyclic cutwidth and linear cutwidth is to expand this concept to complete n -partite graphs, where n represents any number of sets of vertices. Currently a proof has been explored for the linear cutwidth of complete n -partite graphs. The theorem is as follows.

Theorem 10 ([Wei05]). *Let K_{m_1,m_2,\dots,m_n} be a complete n -partite graph with n sets, V_1 ,*

V_2, V_3, \dots, V_n , such that $|V_1| = m_1, |V_2| = m_2, \dots, |V_n| = m_n$. Assume $m_1 \leq m_2 \leq \dots \leq m_n$ and m_1, m_2, \dots, m_n are all even. Then,

$$lcw(K_{m_1, m_2, \dots, m_n}) = \sum_{i=0}^{n-2} \left\lceil \frac{(m_n + \dots + m_{n-i})m_{n-i-1}}{2} \right\rceil$$

There also have been a couple of cases explored for the cyclic cutwidth of complete n -partite graphs. Here the two known cases proved were similar to the two cases for the cyclic cutwidth of complete tripartite graphs $K_{r,s,t}$ where $r \leq s \leq t$ for r, s , and t all even, and $K_{r,s,t}$ where $r = s = t$ for r, s , and t all odd. The theorems are as follows.

Theorem 11 ([All06]). *Let K_{m_1, m_2, \dots, m_n} be a complete n -partite graph with n sets, $V_1, V_2, V_3, \dots, V_n$, such that $|V_1| = m_1, |V_2| = m_2, \dots, |V_n| = m_n$. Assume $m_1 \leq m_2 \leq \dots \leq m_n$ and m_1, m_2, \dots, m_n are all even. Then,*

$$ccw(K_{m_1, m_2, \dots, m_n}) = \frac{m_1 m_2 + m_1 m_3 + \dots + m_1 m_n + m_2 m_3 + m_2 m_4 + \dots + m_{n-1} m_n}{4}$$

Theorem 12 ([All06]). *Let $K_{r, r, \dots, r}$ be a complete n -partite graph with n sets, $V_1, V_2, V_3, \dots, V_n$, such that $|V_1| = |V_2| = \dots, |V_n| = r$. Let r be odd. Then,*

$$ccw(K_{r, r, \dots, r}) = \left\lceil \frac{\binom{n}{2} r^{2+1}}{4} \right\rceil$$

For n even,

$$\frac{\binom{n}{2} r^{2+1}}{4} \leq ccw(K_{r, r, \dots, r}) \leq \frac{\binom{n}{2} r^{2+\frac{n}{2}}}{4}$$

As you can see there are many opportunities for exploration with the cyclic cutwidth of complete n -partite graphs. The next step could be to consider the n -partite graph $K_{r, r, \dots, r, pr}$ where r is odd and p is a natural number.

Bibliography

- [All06] Heather Allmond. On the cyclic cutwidth of complete tripartite and n -partite graphs. CSUSB REU, August 2006.
- [Bow04] Stephanie Bowles. Linear cutwidth of complete bipartite and tripartite graphs. CSUSB REU, August 2004.
- [Hol03] Megan Holben. The cyclic cutwidth of complete bipartite graphs. CSUSB REU, August 2003.
- [Joh03] Matt Johnson. The linear and cyclic cutwidth of the complete bipartite graph. CSUSB REU, February 2003.
- [Wei05] Chelsea Weitzel. The linear cutwidth of complete n -partite graphs. CSUSB REU, August 2005.